# FOUNDATIONS OF GEOMETRIC COGNITION

The cognitive foundations of geometry have puzzled academics for a long time, and even today are mostly unknown to many scholars, including mathematical cognition researchers.

*Foundations of Geometric Cognition* shows that basic geometric skills are deeply hardwired in the visuospatial cognitive capacities of our brains, namely spatial navigation and object recognition. These capacities, shared with nonhuman animals and appearing in early stages of human ontogeny, cannot, however, fully explain a uniquely human form of geometric cognition. In the book, Hohol argues that Euclidean geometry would not be possible without the human capacity to create and use abstract concepts, demonstrating how language and diagrams provide cognitive scaffolding for abstract geometric thinking, within a context of a Euclidean system of thought.

Taking an interdisciplinary approach and drawing on research from diverse fields, including psychology, cognitive science, and mathematics, this book is a must-read for cognitive psychologists and cognitive scientists of mathematics, alongside anyone interested in mathematical education or the philosophical and historical aspects of geometry.

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# PREFACE

The prominent philosopher, logician, and mathematician Bertrand Russell confessed in his *Autobiography* that:

> At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me that it was generally considered difficult, but I had found no difficulty whatever. This was the first time it had dawned upon me that I might have some intelligence. (Russell, 2009, p. 25)

I do not know of many people (in fact, not even one) who could honestly say something like Russell about their first encounter with geometry. Learning geometry from Euclid's original masterpiece, *Elements*, is not that common, either, even at the higher stages of one's formal education. We also do not encounter Euclid's famous fifth postulate, the one that so fascinated the 11-year-old Russell, unless we delve deeper into mathematics. Yet, each of us has encountered Euclidean plane geometry, enriched with some historically more recent inventions, such as a Cartesian coordinate system, at the very earliest stages of our schooling.

Besides arithmetic, most of the contemporary students and laymen alike perceive Euclidean geometry as the prototypical subject of mathematical education. Learning the principles of geometry, in a similar manner to numerical knowledge, plays a pivotal role in the acquisition of mathematical competencies that are useful in everyday life. Euclidean geometry is also extremely significant from the perspective of the history of mathematics. Hellenistic mathematics, including number theory, emerged from the use of geometric concepts and methods. Delving deeper still, the axiomatic-deductive approach to geometry developed by Greeks and depicted in Euclid's masterpiece established a rigorous pattern of the philosophical discourse, or the rational thinking in general, for many centuries. Moreover, many prominent scientists admit that geometric imagination plays a great role in their mathematical thinking. Sir Roger Penrose (2004, 2018), for instance, used to say that he like considering problems geometrically. The same is true of Richard Feynman, as neatly described in his biography by Gleick (2011):

> In high school he had not solved Euclidean geometry problems by tracking proofs through a logical sequence, step by step. He had manipulated the diagrams in his mind: he anchored some points and let others float, imagined some lines as stiff rods and others as stretchable bands, and let the shapes slide until he could see what the result must be. These mental constructs flowed more freely than any real apparatus could. Now, having assimilated a corpus of physical knowledge and mathematical technique, Feynman worked the same way. The lines and vertices floating in the space of his mind now stood for complex symbols and operators. They had a recursive depth; he could focus on them and expand them into more complex expressions, made up of more complex expressions still. He could slide them and rearrange them, anchor fixed points and stretch the space in which they were embedded. Some mental operations required shifts in the frame of reference, reorientations in space and time. The perspective would change from motionlessness to steady motion to acceleration. (p. 161)

These insights correspond with the frequently cited, and at the same time highly controversial, observation of Henri Poincaré (1929) that mathematicians can be divided into two camps: analysts and geometers, wherein it "does not prevent the one sort from remaining analysts even when they work at geometry, while the others are still geometers even when they occupy themselves with pure analysis" (p. 210; see Hadamard, 1945). All of this makes geometric thinking a fascinating topic.

Like any human intellectual enterprise, Euclidean geometry also emerges from cognitive processes and the activity of our brains. Nevertheless, the cognitive origins of geometry remained puzzling for a long time. A cognitive revolution that occurred in the mid-1950s, together with the later development of a new discipline called cognitive science, made this subject explorable in a scientific way. Nevertheless, the foundations of geometric cognition remain mostly unknown even today for the majority of mathematicians, historians of science, educational researchers, philosophers, psychologists, and cognitive scientists. Furthermore, even those of the latter who are interested in mathematical cognition focus primarily on the cognitive processing of numbers and calculations. This is manifested in the fact that the problem of geometry is essentially absent in most of the fundamental monographs in the field of mathematical cognition. We can observe a similar pattern in the cases of scientific conferences and journals. In contrast to the processing of numbers, there is no cyclical conference or peer-reviewed journal specializing in geometric cognition.

Although I have decided to limit my investigation in this book to Euclidean geometry, developed in ancient Greece but still taught to some degree to children today, starting in primary school, identifying crucial properties of geometric cognition remains a difficult task.[1] Despite the fact that children and the majority of educated adults are unfamiliar with proving theorems in an axiomatic-deductive fashion, leaving this method of reasoning aside entirely would make my investigation grossly incomplete. The proof is "a hard core" of Euclid's contribution to the whole of mathematics. According to this fact, I try to explain not only where elementary manifestations of geometric cognition, such as sensitivity to angle, length (distance), and sense (left-right direction) come from, but also how the processing of abstract geometric concepts works and how Euclidean proofs that provide general results in a necessary way are cognitively possible at all. My proposal for sketching the account of the cognitive foundations of Euclidean geometry involves the following desiderata:[2]

(D1) The account should recognize whether the cognitive capacities that are necessary to engage with Euclidean geometry are "hardwired," or whether they are rather constructed through individual learning.
(D2) The account should describe how these capacities are combined during ontogeny into a system of abstract geometric concepts.
(D3) The account should elucidate how the mind/brain of the human being (especially if it is constrained by the body and environment) is able to process abstract concepts at all.
(D4) The account should elucidate the geometric proof characterized by epistemic virtues: compelling power (or necessity) and generality of providing results.

Let me briefly explain how I intend to explore the above desiderata in the following chapters. In Chapter 1, I investigate different perspectives on geometric thinking, involving the history of mathematics, philosophy, early experimental psychology, education research, and, finally, interdisciplinary cognitive science, which will be further explored in subsequent chapters. This review chapter, which is the most extensive, will familiarize the reader with the crucial notions of Euclidean geometry and existing approaches to mathematical cognition, as well as the research problems that are particularly associated with the desiderata (D1–D4).

In Chapter 2, I attempt to identify the "hardwired" foundations of geometric cognition, namely those occurring in human beings at an early developmental stage and shared with nonhuman animals. Adopting Tinbergen's strategy of explanatory questions I show that the sensitivity to elementary Euclidean properties is not a uniquely culture-dependent human skill that emerges when learning geometry in school. This sensitivity is observed in the context of the recognition of shapes and spatial navigation in many animal species and human infants in many cultures. Therefore, I defend a version of the "hardwiredness" of elementary geometric cognition (D1), elaborating upon this notion by grounding it in the findings of various branches of cognitive science and related fields as cognitive and developmental psychology, neuroscience, evolutionary biology, ethology, comparative cognition, and behavioral robotics. Bearing in mind that only humans have developed, and are capable of assimilating, full-blooded Euclidean geometry, in this chapter I also investigate the limits of hardwired geometry and how children go beyond them. Thereby, I show that the process of the acquisition of the abstract conceptual structures of geometry is first mediated at the preschool level by enculturation with spatial language and map-like scale models (D2).

It is a truism to say that the concepts of Euclidean geometry are abstract in nature (D3). On the other hand, there is much evidence to show that the body and physical surroundings constrain the thinking of human beings. Therefore, in Chapter 3, I investigate how the processing of abstract concepts, something which seems to require us to reach beyond our proximal experience, is at all cognitively possible. I start by discussing the classic view of computational cognitive science on conceptual processing and show that it faced severe challenges (e.g., the symbol grounding problem) that stimulated cognitive science to shift the research paradigm. In this context I describe the emergence of embodied cognitive science while simultaneously claiming that in its strong version, where it assumes that the sensorimotor system of the brain both serves as the conceptual vehicle and determines the conceptual content, it is ill suited for the elucidation of geometric abstracts. I defend the claim that (D3) can be fulfilled by adopting the moderate version of embodiment, one that makes room for the shaping of the content of abstract concepts by internalizing natural language. Following Lev Vygotsky and contemporary theorists of the moderate version of embodied cognitive science, I show that, by virtue of its social nature and computational properties, human language serves as scaffolding for further learning. In other words, it is a cognitive artifact that makes establishing and using abstract concepts cognitively possible.

The above-summarized chapter does not directly answer the question of the origin of "the power of proof," or the epistemic virtues of Euclidean geometry. Therefore, in Chapter 4, I seek a cognitive base for the compelling power (or necessity) of Euclidean reasonings and the generality of provided results (D4). To this end I shift the perspective of my investigation from experimentally oriented cognitive science to the cognitive history of geometry as developed by Reviel Netz. I make use of the notion of the cognitive artifact introduced in Chapter 3 and trace the role of two intertwined inventions of the ancient Greeks, namely lettered diagrams and well-regulated professional language, which helped to build a cognitive niche within which the necessity and generality of geometric proofs became possible. In this context, I hypothesize that the professional language of geometry—one that is mutually interconnected with diagrams through letters associated with geometric points—is characterized by the computational properties enhancing the hardwired cognitive capacities of the human being to a greater extent than ordinary or everyday language. At the end of the book, I list the limits of my investigation and outline the perspectives for further research on geometric cognition.

## Notes

1. Cognitive scientists sometimes do not perceive geometry as a phenomenon to be explained, or explanandum, but they use geometric or topological structures for modelling cognitive processes and representations. Geometry is used to explain mental phenomena, or plays a role in the explanans, for example, in Gärdenfors' (2004, 2014) theory of conceptual spaces.
2. Note that the following list is a modified version of one introduced previously in our article (Hohol & Miłkowski, 2019). The current proposal is undoubtedly not final, and does not pretend to be complete. Assuredly, the desiderata for the study of geometric cognition will change during the evolution of the cognitive science of mathematics and related fields.

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Ultimately, this work is the fruit of my infinite love for my wife, a brilliant soulmate and inspiring researcher who taught me to ask all of the right questions: Kinga Wołoszyn-Hohol.

Mateusz Hohol
*Kraków, May 2019*

# 1

# GEOMETRIC THINKING, THE PARADISE OF ABSTRACTION

## 1.1 Introduction and synopsis of the chapter

The purpose of this chapter is to provide an overview of different perspectives on geometric thinking. My ambition is not only to tell the story (actually very incomplete) of the various faces of geometric thinking, including perspectives of the history of mathematics, philosophy, early experimental psychology, education studies, and finally interdisciplinary cognitive science, but also to establish "a searching space," which allows us to identify some crucial aspects of geometric cognition on the basis of the achievements of different (but related) perspectives. My further goal is also to familiarize us with terms that we will meet in the next chapters.

The chapter proceeds as follows. First (Section 1.2), I will introduce the roots of geometry, showing that Euclid's *Elements* is founded on the efforts of generations of mathematicians who brought geometry to the paradise of abstraction. Subsequently, in Section 1.3, I will show that the geometric intuition is *a locus classicus* of European philosophy stretching from Plato to Helmholtz. The considerations of the latter—Helmholtz—were scientifically inspired while still philosophical in nature. In Section 1.4, I will tackle the first full-blooded experimentally oriented account of the development of geometric cognition, as elaborated by Piaget and Inhelder. Following this, I will still focus on developmental issues, but now perceived from the educational perspective. Thus, in Section 1.5, I will introduce and discuss the classic model of the emergence of geometric skills introduced by van Hiele and van Hiele-Geldof. In Section 1.6, I will present the cognitive revolution that allowed for the development of a broader research perspective on geometric cognition, one that will serve as our companion throughout the remainder of the book. Discussing cognitive science studies on mathematics, I will show that during its own evolution, this field shifted toward numerical cognition and left Euclidean geometry by the wayside. Section 1.7 is a summary.

## 1.2 The geometric roots of mathematical thinking

Euclid's splendid masterpiece, the *Elements*, unquestionably set the agenda of the subsequent development of European mathematics and established the pattern of mathematical rigor for many centuries to come. Euclid's system did not, however, emerge from a vacuum, but summarized, or rather developed further, the achievements of his predecessors. Thus, prior to discussing it, we should introduce the distant, pre-Euclidean origins of geometry.[1] At the very beginning, however, we have to stipulate that if we accept the broadest possible, and at the same time very intuitive, definition of geometry, which states that it is *the science of space*, the times to which we withdraw did not contain anything akin to institutional science. Instead, the comprehension of space was a very practical enterprise; that is, it was connected with architecture and land measurement. In this sense, one can say that the history of geometry begins before geometry.

The oldest material premises for the use of geometric regularities by man date back to the earliest megalithic cultures (Dzbyński, 2014). The ancient Egyptian constructions that are partially preserved to this day, such as the temple of Abu Simbel or the pyramids, as well as Babylonian ziggurats, reveal a sophisticated sensitivity to geometric form and allow us to suppose that their constructors were familiar with at least some rudimentary geometric knowledge (O'Leary, 2010). There are also historical premises that Egyptians were highly proficient in techniques for land measurement involving knowledge about geometric relationships. One of the earliest mentions concerns the determination of the amount of land tax in relation to the annual flooding of the Nile. Herodotus (2009), the fifth-century BC Greek historian, described the following story of nineteenth-century BC pharaoh Sesostris, also written as Senusret III, who reigned in Egypt during the Middle Kingdom period:

> This king distributed the land to all the Egyptians, giving an equal square portion to each man, and from this he made his revenue, having appointed them to pay a certain rent every year: and if the river should take away anything from any man's portion, he would come to the king and declare that which had happened, and the king used to send men to examine and to find out by measurement how much less the piece of land had become, in order that for the future the man might pay less, in proportion to the rent appointed: and I think that thus the art of geometry was found out and afterwards came into Hellas also. (Book II, p. 109)

Although it is far from clear whether the story is true—Herodotus stated that the priests of Thebes had informed him that it was the case—the Egyptian origins of geometry are also mentioned in other Greek historical sources. The story related by Herodotus reveals the source of the Greek term γεωμετρία, which literally means the measurement of the Earth. Before we consider how geometry was transmitted from Egypt to Greece, as the Greek historian suggests, it is worth emphasizing that Egyptian knowledge about figures or polyhedrons was strictly technical and took the form of practical rules (Merzbach & Boyer, 2011). Applying these rules to particular problems did not lead the Egyptians to invent abstract mathematics with the necessary and general arguments characteristic of Euclidean geometry. Instead, as Russo (2004) has summed up, Egyptian achievements

> can be called mathematical only in that their object is solving problems that we would call arithmetical or geometric; they completely lack the rational structure that we associate with mathematics today. They contain recipes for solving problems—for example, calculating the volume of a truncated pyramid or the area of a circle (the latter being, of course, unintentionally approximate)—but there is no sign of anything like a justification for the rules given. At that stage, then, fairly elaborate notions beyond the integers had already been developed, including many plane and solid figures, area, and volume; problem-solving methods were passed down the generations; but the correctness of the solutions was based solely on experience and tradition. This was very far from being a science in the sense we have given the word. It was simply a part of that enormous store of empirical knowledge that enabled the Egyptians to achieve their famous technological feats; it was methodologically homogeneous with the rest of such knowledge, and transmitted in the same way. (p. 32)

According to the traditional view, the art of geometry came to Greece together with Thales of Miletus, called the first Greek philosopher, who lived at the turn of the seventh and sixth century BC (see Aufrere & Marganne, 2019; O'Grady, 2002). Because his writings have not survived, this opinion is based on later documents, such as a treatise by the fifth-century Neoplatonic philosopher Proclus, who refers to the now-missing history of mathematics by Eudemus of Rhodes (fl. ca. 320 BC). According to this tradition, Thales traveled to Egypt, where he became familiar with a technique for determining the height of pyramids on the basis of the measurement of their shadows. He also visited Babylon, where he learned a proposition of elementary geometry which stated that when A, B, and C are distinct points on a circle and the line AC is a diameter, then the angle ABC is a right angle, something which is known today as Thales' theorem (see Figure 1.1). Proclus (1970) claims, however, that Thales also made original contributions to geometry: "He made many discoveries himself and taught his successors the principles for many other discoveries, treating some things in a more universal way, others more in terms of perception" ([8.74], 65.7–11).

The discoveries mentioned by the Neoplatonic philosopher concern the following geometric facts: "a circle is bisected by a diameter," "the base angles of an isosceles triangle are equal," "the pairs of vertical angles formed by two intersecting lines are equal," and "if two triangles are such that two angles and a side of one are equal, respectively, to two angles and a side of the other, then the

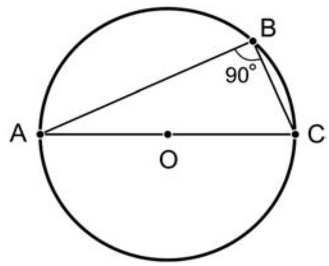


**FIGURE 1.1** Thales' theorem. According to Thales' theorem, if A, B, and C constitute distinct points on a circle where the line AC is a diameter, then the angle ABC is a right angle.

triangles are congruent." The most impressive contribution to geometry that is traditionally attributed to Thales seems to be, however, "treating some things in a more universal way." The point is that he allegedly not only knew the above facts, but he was also supposed to demonstrate them deductively. Simultaneously, Thales supposedly introduced a principle that similar figures, regardless of their material substance, have the same geometric properties. Thus, Thales of Miletus is sometimes designated not only as the first Greek philosopher but also the first full-blooded geometer. One must be aware, however, that this claim is of a semilegendary nature.

Although Thales is portrayed as the first geometer, the tradition says that Pythagoras, the prophet, mystic, and philosopher of Samos (6th–5th century BC), established the first Greek mathematical school (see Kahn, 2001). He presumably trod the same well-worn road to Egypt and Babylon where, like Thales, he was able to learn the art of geometry firsthand. Transferring these mathematical ideas to Greece, he gave them an abstract and general form. One of the essential achievements attributed to him is the discovery of the theorem, known today as Pythagoras' theorem, that the square built on the triangle's side opposite to a right angle is equal to the sum of two squares formed on the other two sides of the triangle (see Figure 1.2). Even though the geometric relationship was imported rather than discovered by Pythagoras, neither Egyptian or Babylonian geometers were aware of its proof, while the philosopher of Samos was supposed to be the first to demonstrate it. In a more general perspective, as Proclus (1970) said, "Pythagoras transformed the philosophy of geometry into the form of a liberal education, searching in an upward direction for its principles and investigating its theorems immaterially and intellectually" ([8.74], 65.7–66.8). In a manner akin to Thales, Pythagoras' contribution to geometry also has semilegendary character.

Contemporary historians agree that Pythagoras established a school or something akin to a secret association in Croton around 530 BC, but its character was initially religious. One of the central points of Pythagorean doctrine was number mysticism, a connected conviction that numbers constitute "the elements

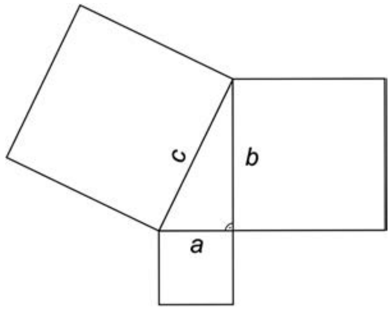


**FIGURE 1.2** Pythagoras' theorem. According to Pythagoras' theorem, the sum of the areas of the two squares built on the legs *a* and *b* is the same as the area of the square built on the triangle's side *c* opposite the right angle.

of all things" (Aristotle, 2009a, p. 986a). The discovery of incommensurability undermined this conviction, triggering both a religious and mathematical crisis (Knorr, 1974). It turned out that there are magnitudes, for example, lengths of a diagonal of a square, characterized by irrational numbers. The crisis was overcome by means of the deployment of geometric algebra, namely replacing numbers and numerical operations with geometric figures such as line segments, rectangles, and parallelepipeds. We will see more of the technical aspects in the later part of this section, but for now let us emphasize the fact that, in resolving the crisis of incommensurability, the Pythagoreans transformed geometry into a fundamental branch of Greek mathematics.

The detachment of geometry from "concrete things" did not happen overnight, but instead took the form of a continuous process. Furthermore, instead of being the achievement of a single person, it should be assigned to the disciples and followers of Pythagoras as well. Even though the scientific part of the secret community that focused primarily on mathematics emerged relatively late, the initial treatment of mathematics as a spiritual exercise and component of worship could help encourage, or even drive, the abstract investigation of mathematical relationships and, consequently, conducting demonstrations of geometric theorems (A. Seidenberg, 1961). It is also possible that the use of specific tools or artifacts played a crucial role in the emergence of abstraction and deduction in Greek philosophy.

The contribution of the Pythagoreans comprises the introduction of two of the tools used in Euclidean geometry to this day, namely elements of technical language and a special kind of drawings. Regarding the former, the Pythagoreans introduced a few well-known concepts for first time such as "straight line," "line segment," "plane," and "angle." It was also noteworthy that they distinguished between various kinds of angles, for example, "inscribed angle" and "central angle," and defined mathematical terms. Regarding the latter tool, drawings of geometric shapes, known to us today as diagrams, were undoubtedly used earlier, and we can find them in various cultures throughout the ancient world (notably, the Greek term διάγραμμα literally means "figure marked out by lines" [cf. Netz, 1999b, p. 35]). However, Pythagoreans presumably invented scale drawings and understood that increased or decreased figures might be similar, holding the same geometric properties. Moreover, at least according to tradition, Hippocrates of Chios, a fifth-century BC Greek mathematician who belonged to the Pythagorean school before being expelled for the terrible crime of teaching for money, was the first to use letters to mark points on geometric diagrams (Merzbach & Boyer, 2011, p. 61). I will return to technical language and lettered diagrams in Chapter 4, where I will try to interpret them as cognitive artifacts, namely tools affecting, or even shaping, geometric cognition.

As we have seen, Thales and Pythagoreans transformed the art of the measurement of concrete objects and fields into the science on space, which uses abstract concepts and demonstrates general theorems. As Merzbach and Boyer (2011) say, "with them, mathematics was more closely related to a love of wisdom than to the exigencies of practical life" (p. 45). Although the first deductive reasoning is credited to Thales and Pythagoreans, they were certainly not familiar with the axiomatic-deductive method, according to which theorems should be derived from a set of well-defined axioms (or postulates) under the rigor of necessity-preserving rules. The axiomatic-deductive method originated, however, outside the science of space, having been initiated by Aristotle.

Aristotle was no mathematician, but his contribution to logic and scientific methodology strongly influenced the development of geometry (see Heath, 1970). In his *Posterior Analytics*, Aristotle (2009b) stated that all reliable knowledge consists of two kinds of true propositions (statements): self-evident ones, which do not require further justification, and propositions demonstrated "by showing that it is a logical consequence of propositions already known to be true" (Murawski, 2010, p. 41). According to Aristotle, each theorem contains components of several kinds: *the definitions* of the terms introduced in a theory; *the principles*, assumed without a proof (some of them, called "the axioms," characterize the fundamental properties of magnitudes, and others, called "postulates," refer to the entities studied by the specific discipline); and, last but not least, *existential statements* postulating the existence of objects specified by a theorem. These components, as Aristotle claimed, should be used in demonstrations, namely chains of immediate inferences where propositions are transformed without losing their truth value.

The axiomatic-deductive method that was modeled on the Aristotelian idea was incorporated into geometry on the largest scale in the Greek world by Euclid (Mueller, 1981; Murawski, 2010). The geometric reasonings enshrined in his *Elements* were recognized as a model of intellectual rigor and a prototype of scientific thinking for many centuries, mainly as a result of two epistemic virtues: these reasonings are necessarily true and lead to universal results. We know, however, very little about the life of the author of *Elements*. Frankly, we do not even know when and where exactly he was born and died. Frequently it is assumed that he lived between 325 and 270 BC and studied at Plato's Academy, or at least under the supervision of one of Plato's apprentices—according to Proclus (1970), "Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures" (p. 68). The author of *Elements* is called Euclid of Alexandria since he worked in this city under the reign of Ptolemy I Soter. The tradition says that Ptolemy asked Euclid to indicate a shorter way to understand geometry than through studying the whole of the *Elements*. The geometer apparently answered, as Proclus (1970) reported, "there was no royal road to geometry" (p. 68). Even though we do not have many biographical details about Euclid, we have at our disposal more of his treatises than of any other Greek mathematician (Merzbach & Boyer, 2011). In addition to *Elements*, the following treatises by Euclid have survived to our times: *Data* (the content is interpreted as supplementary material to a couple of first books of *Elements*), *On Division of Figures* (as the title suggests, it concerns the division of plane configurations into parts), *Phenomena* (on spherical astronomy), and *Optics* (on the geometry of direct vision). Let us look at the structure of the content of Euclid's greatest work.

*Elements* consists of 13 books or—to say more modernly—chapters.[2] The first six books introduce elementary plane geometry and geometric algebra. To briefly illustrate how geometric algebra works, let us introduce Proposition 1 of Book 2, which states that "if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line)" (I use the translation of Euclid's *Elements* by Fitzpatrick, 2008). This assertion is a geometric equivalent of the distributive law, which is today expressed in the following form:

$$a(b + c + d) = ab + ac + ad$$

Let us introduce further books. The next three concern the geometric theory of numbers, according to which each natural number is conceptualized by a line segment and multiplication is represented in terms of measuring. The 10th book covers the problem of incommensurables, and the final three books deal with the geometry of Platonic solids.

Book 1 opens with a list of 23 definitions (further definitions are introduced in the following books). The first three define "a point," "a line," and "the extremities of a line": "1. A point is that of which there is no part," "2. And a line is a length without breadth," and "3. And the extremities of a line are points."[3] Immediately after definitions, Euclid lists principles that involve two sets: five postulates and five common notions.[4] After them, the author presents geometric theorems as so-called "propositions." They involve constructions that allow the reader to understand geometric relationships or—according to the other interpretation—bring these relationships into being. I will return to this issue in the next section. For now, let us list "a methodological skeleton" of Book 1 of *Elements*, namely all of the postulates and common notions.

Postulates:

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

Common notions:

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

We do not have room for a detailed analysis of the content, origins, and importance of all of Euclid's initial statements. It suffices to note that the vast majority of postulates and common notions are well grounded in tradition, transparent or even self-evident, and easy to grasp—one can "see" them quickly—and thus they did not raise reservations either in ancient Greece or in later times. The situation is dramatically different in the case of the famous fifth postulate, also known as the parallel postulate. Since the postulate seemed not to be self-evident and instead rather complicated, Proclus (1970), who did not deny its truth, claimed that "this ought even to be struck out of the Postulates altogether" (p. 150). For two millennia, successive generations of mathematicians made attempts to prove the fifth postulate using the remaining four ones; however, these efforts turned out to be unsuccessful. In 1868, Italian mathematician Eugenio Beltrami finally demonstrated the independence of the postulate from the others listed in *Elements* (see Bardi, 2008 for an accessible introduction). Furthermore, in the first half of the nineteenth century, János Bolyai, Nikolai Ivanovich Lobachevsky, and Carl Friedrich Gauss independently discovered the possibility of construction of logically consistent geometric systems by replacing the parallel postulate with other ones. This discovery paved the way for the development of so-called non-Euclidean geometries, that is, hyperbolic and elliptic geometry.[5]

Let us return to Ancient Greece and try to determine the scientific status of Euclid's *Elements*. On the one hand, it is called a mathematical treatise, which, thanks to the author's original discoveries, established or founded full-blooded geometry. On the other hand, we know well that *Elements* served as a mathematical textbook already during the author's lifetime and for about two millennia afterward. There is virtually no contradiction in perceiving Euclid's work both as a treatise and a textbook (note that the distinction between these writing forms is new); however, according to George Sarton, we should avoid two extreme interpretations. The first interpretation, according to Sarton (1959), speaks about Euclid as the originator or founding father of geometry:

> If we take Egyptian and Babylonian efforts into account, as we should, Euclid's *Elements* is the climax of more than a thousand years. One might object that Euclid deserves to be called the father of geometry for another reason. Granted that many discoveries were made before him, he was the first to build a synthesis of all the knowledge obtained by others and himself and to put all the known propositions in a strong logical order. That statement is not absolutely true. Propositions had been proved before Euclid and chains of propositions established (…). In short, whether you consider particular theorems or methods or the arrangement of the *Elements*, Euclid was seldom a complete innovator; he did much better and on a larger scale what other geometers had done before him. (pp. 23–24)

The second interpretation depicts Euclid only as "*a textbook maker* who invented nothing and simply put together in better order the discoveries of other people" (p. 24). In Sarton's opinion, both interpretations—Euclid as an originator of geometry and Euclid as a textbook writer—are mistaken, since, as he continues:

> A good many propositions in the *Elements* can be ascribed to earlier geometers, but we may assume that those which cannot be ascribed to others were discovered by Euclid himself; and their number is considerable. As to the arrangement, it is safe to assume that it is to a large extent Euclid's own. He created a monument which is as marvelous in its symmetry, inner beauty and clearness as the Parthenon, but incomparably more complex and more durable. (ibid, p. 24)

According to Sarton (ibid., pp. 24–36), Euclid's original discoveries enshrined in *Elements* involve, at least, formulating the famous fifth postulate, which—as we have already seen—stimulated mathematical investigations for two millennia; introducing several theorems of number theory, such as the existence of infinitude of primes; and formulating the fundamental laws of geometric optics, to list only the law of reflection.

The uniqueness of Euclid's genius lies also, or perhaps primarily, in the fact that he was capable of constructing both a versatile and a homogeneous "logical space." It is versatile, since it not only covers elementary plane geometry and solid geometry, but also such mathematical fields as algebra and theory of numbers. It is simultaneously homogeneous since all of these fields are comprehended in geometric terms. Thanks to the versatility and homogeneity of Euclid's masterpiece, as the Dutch mathematician Hans Freudenthal (1971) said, "for a long time mathematics has been synonymous with geometry" (p. 417). *Elements* has undoubtedly affected the mathematical thinking of successive generations, and its impact extends beyond mathematics—the discussion on the sources of Euclidean geometry is a classical theme in European philosophy. This debate has worn various masks from the ontological to the epistemological. In the next section, we will consider the recurring theme of these discussions, namely intuition as a source of geometric knowledge.

## 1.3 Geometric intuition as a philosophical locus classicus

"Let no one ignorant of geometry enter herein" (ἀγεωμέτρητος μηδεὶς εἰσίτω). According to tradition, this motto was engraved at the entrance to Plato's Academy, and it expresses the notion that mathematics alone delivers a necessary prerequisite of philosophy: training in abstract thinking. This common interpretation goes hand in hand with the traditional account that Plato was a highly demanding master who required from his philosophical apprentices the study of geometry for at least 10 years, while the course of philosophy lasted only 5. Although he lived before Euclid, already in Plato's lifetime (approximately 427–347 BC), geometry was a sophisticated matter, and adepts had to put in a lot of effort to achieve mastery.

Plato did not, however, claim that recognizing all geometric truths is solely the merit of extensive training. On the contrary, he believed that man is capable of intuitively and effortlessly grasping rudimentary knowledge on geometric points, line segments, angles, and relationships between figures. Plato (2009a) illustrates it in one of the Socratic dialogues entitled *Meno*, where the uneducated slave boy conducts deductive reasoning that leads to the necessary conclusion that a square built on the diagonal of a given square is double. The demonstration takes into account that the bigger square is formed of four triangles, wherein each of triangles is equal to half of the given square (Figure 1.3).

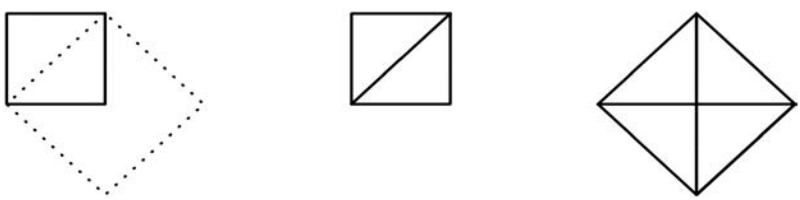

**FIGURE 1.3** The geometric proof by Plato's slave boy. The figure is patterned after Russo (2004, p. 37) and visualizes how the geometrically naïve slave boy described by Plato in *Meno* proved that the square built on the diagonal of a given square is double. His informal reasoning refers to the fact that the bigger square is formed of four triangles, wherein each of triangles is equal to half of the given square.

Although Plato was fully aware that achieving mathematical mastery requires long-term training, he believed that the human being is equipped with an "insight," or "intuition,"—an innate, or hardwired, sense of mathematical ideas that constitute a foundation of further practice and it is not an outcome of "habit, practice or convention" (see Parsons, 1980, p. 146). Thanks to the intuition, each adept of geometry (such as the slave boy) is capable not only of conducting simple deductive reasoning but also of perceiving the self-evident truthfulness of mathematical axioms (postulates and common notions). Such a view is complemented by Plato with a claim expressed in *The Republic* (Plato, 2009b) that "the knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient" (527b). The knowledge, accessible through Platonic intuition, is persistent, unchanging, and necessary, and therefore it is discovered, rather than being invented or arbitrarily created by man (see Wedberg, 1955, pp. 63–82). According to Plato, the geometric objects constructed by mathematicians by means of diagrams and linguistic expressions are not appropriate objects of geometric knowledge. As Detlefsen (2005) has summed up the discussion, "they could at best serve as *representations* of real objects and provide some sort of practical guide to their knowledge" (p. 243).

Although Euclid was educated in the Academy, or at least had contact with Platonism, in his *Elements*, as well as other treatises, he avoided direct philosophical declarations, and thus we do not know much about his views on the prerequisites of geometry and the meaning of geometric constructions. We do know, however, that these issues were extensively discussed by Greek philosophers, such as Proclus, who was influenced both by Plato and Euclid. This fifth-century BC Neoplatonist called the Successor was one of the last heads of Academy (see D'Hoine & Martijn, 2017 for an overview) and the author of *A Commentary on the First Book of Euclid's Elements* (1970)—a treatise to which we have referred many times and will do so further—which refined, or developed further, the Platonic philosophy of mathematics (O'Meara, 2017).

Referring to other Platonists, namely Speusippus and Amphinomus, Proclus stated that the construction of geometric objects is not in the making of them, but rather understanding them, that is, "taking eternal things as if they were in the process of coming to be" (1970, p. 78; see Bowen, 1983). In other words, constructions allow man to grasp what has always existed. Proclus believed that perfect geometric objects and their relationships cannot be derived or abstracted by man from their imperfect and deficient material shadows. Consequently, he said that Euclidean constructions are possible thanks to the contemplation of intelligible geometric principles that are innate, albeit initially hidden, in the soul. As O'Meara (2017) notes:

> Mathematics starts from this innate knowledge, developing it on the level of articulated logical reasoning (discursive thought) in the form of concepts which are defined and propositions (axioms) which are stated. These first articulations of innate knowledge are then combined so as to deduce their

> consequences, i.e., the conclusions that can be derived from them. Mathematics is thus a "projection" in discursive thought of the innate knowledge of soul. Mathematical objects both constitute soul, as intelligible principles, and are constituted by soul, as the concepts, propositions, and arguments which soul elaborates (or "unrolls," a favourite image) by rational methods from these principles. (p. 172)

This does not mean, however, that all Hellenistic philosophers agreed with the Platonic claim that the foundations of geometric knowledge are innate. Aristotle, for example, stated that mathematical entities are intellectually abstracted from physical objects, and thus geometric knowledge has an empirical character (Heath, 1970). The same is true regarding the status of geometric constructions: not everyone agreed that geometric constructions are representations, or reflections, of eternal geometric objects. The fourth-century BC mathematician Menaechmus, a personal friend of Plato and an associate of the Academy, interpreted geometric constructions entirely literally, namely as "a process by which objects were produced or generated" (Detlefsen, 2005, p. 244). Simultaneously, Menaechmus rejected the view that the matter of mathematics is the contemplation of eternal forms (see Bowen, 1983).

As we have seen, the philosophical debate about the status of Euclidean geometry was already raised in Ancient Greece and later continued in subsequent epochs. Although we are unable to examine this in detail, a few of its modern episodes may be illustrative at this point. René Descartes was both the founder of modern philosophy and a creative mathematician. As a mathematician, Descartes (1637/2012) launched a new field: an algebraic geometry, which applied established seventeenth-century algebra to the treatment of geometric problems (see Lenoir, 1979). As an influential philosopher, he further grounded geometry as the standard for all rational discourse, claiming that geometric theorems reveal prototypical "eternal truths" (Descartes, 1976). Intuition, also called "natural light," was the cornerstone that allowed our mind's eye to perceive the "clearness and distinctness" of mathematical theorems (Morris, 1973). According to Descartes (1684/1998), intuition is not "the fluctuating testimony of the senses" nor "the deceptive judgment of an imagination which composes things badly" (p. 79). Instead, intuition is

> the conceptual act of the pure and attentive mind, a conceptual act so easy and so distinct that no doubt whatsoever can remain about what we are understanding. Alternatively, it amounts to the same thing to say that by "intuition" I understand the indubitable conceptual act of the pure and attentive mind, which conceptual act springs from the light of reason alone. Because this act is simpler, it is more certain, than deduction, which, however, as we have noted above, a human being also cannot perform wrongly. Thus everyone can mentally intuit that he exists, that he is thinking, that a triangle is bounded by only three lines, that a sphere is

> bounded by a single surface, and similar things, which are much more numerous than most might realize, since they disdain to turn their minds to such easy matters. (ibid., pp. 79–80)

Finally, it should be emphasized that although Descartes understood intuition as a purely intellectual capacity, or something belonging to the sphere of thinking, he conceptualized intuition according to a traditional metaphor, stating that "thinking is seeing."

The issues of mathematical intuition and sources of geometric knowledge were especially crucial for Immanuel Kant. We should recall his views briefly here since they enjoyed a wide impact on interpreting Euclidean geometry, as a "privileged" (in relation to others) mathematical system. The Königsberg philosopher claimed that all mathematical theorems, including the first principles, have the status of so-called *synthetic a priori propositions*. "Synthetic," according to Kant, means that a proposition's predicate concept is not contained in its subject concept, and therefore a proposition expands our knowledge; "a priori" means that the truth of a proposition is independent of empirical justification and may be recognized intuitively.[6] In the *Prolegomena to Any Future Metaphysics*, Kant (1783/2004) noted that mathematical reasoning:

> carries with it thoroughly apodictic certainty (i.e., absolute necessity), hence rests on no grounds of experience, and so is a pure product of reason, but beyond this is thoroughly synthetic … All mathematical cognition has this distinguishing feature, that it must present its concept beforehand *in intuition* and indeed *a priori*, consequently in an intuition that is not empirical but pure, without which means it cannot take a single step; therefore its judgments are always *intuitive*. (p. 32)

In other words, mathematical propositions, such as "all rectangles have four sides as well as four right angles," appear to be justified purely intuitively, but they are not tautological since they deliver information beyond that encoded in a subject concept.

Kant claimed that mathematical propositions are simultaneously synthetic and a priori, since they are associated with transcendental structures organizing cognition, namely representations of space and time (Brożek & Olszewski, 2011; Parsons, 1980). Notably, he understood space and time not as independent substances (things), or something external to the human being, but instead as internal components of the mind (or the transcendental ego, in Kantian terms): the forms that filter our sensual impressions. The internal representation of space allows us to construct geometry, while the representation of time underlies the construction of arithmetic. As Kant (1783/2004) said:

> Now space and time are the intuitions upon which pure mathematics bases all its cognitions and judgments, which come forward as at once apodictic

> and necessary; for mathematics must first exhibit all of its concepts in intuition—and pure mathematics in pure intuition—that is, it must first construct them … Geometry bases itself on the pure intuition of space. Even arithmetic forms its concepts of numbers through successive addition of units in time, but above all pure mechanics can form its concepts of motion only by means of the representation of time. (p. 35)

From its ancient beginnings, geometry was a science about space that we now call Euclidean. As mentioned in the previous section, this state only changed in the first half of the nineteenth century, when coherent geometric systems with the negation of the parallel postulate were developed. Kant perceived Euclidean geometry not only as a privileged field of mathematics but also stated that the space understood as the inner component of the human mind (resp. transcendental ego) is a starting point for the construction of geometry, which is necessarily Euclidean. In other words, he believed that the intuition leads us directly to Euclidean structures. Geometry does not come, however, from the passive contemplation of space. Kant emphasized the active role of the human mind in the construction of geometric entities. Brożek and Olszewski (2011) nicely summarized this aspect of Kant's doctrine: "mathematical knowledge cannot be gained by recourse to concepts. The analysis of pure concepts cannot lead us to the establishment of any mathematical theorem—in order to prove anything, we need to construct our concepts, and for the construction we need intuition" (p. 89).

Let us try to evaluate Kant's contribution to the philosophy of geometry. On the one hand, historians of mathematics often stress that Kant's unquestionable authority inhibited the reception of non-Euclidean geometries for decades and contributed to the absolutization of the Euclidean system, namely perceiving it as the matter of truth. On the other hand, Kant's concept of the link between geometric intuition and the internal representation of space has inspired generations of researchers who, nevertheless, mostly disagreed with his claim that intuition can only lead to geometric constructions constrained by the set of Euclidean postulates. In the final paragraphs of this section I will take a look at two prominent figures—Henri Poincaré and Hermann von Helmholtz—who were strongly influenced by Kant but rejected his claim that Euclid's postulates are indisputable.

Henri Poincaré, the French scientist of the turn of the nineteenth and twentieth centuries, is known for his tremendous impact on numerous disciplines, such as pure mathematics (e.g., he was one of the cofounders of topology and the author of the conformal disk, which is a model of hyperbolic geometry), mathematical physics (almost simultaneously to Einstein, he developed the mathematical foundations of special relativity theory), and philosophy, particularly the general methodology of science, as well as the philosophy of mathematics (see J. Gray, 2012). Mathematical intuition was one of his main interests (Murawski, 2004). Inspired by Kant, Poincaré (1905) perceived it as the innate creative power of the human mind. This capacity fosters the construction of mathematical concepts and makes that mathematical theorems are perceived as clear and distinct. Furthermore, Poincaré considered generalization by induction, which allows the formulation of synthetic mathematical propositions—namely those that expand our knowledge—a manifestation of aprioristic intuition. He also considered the relationship between conscious and unconscious levels of mathematical processing. According to Poincaré, a large number of mathematical theorems originate at the unconscious level, but must be completed by consciously controlled reasoning.

Although Poincaré referred explicitly to Kant, he developed an original position in the field of the philosophy of geometry. In contrast to Kant, Poincaré stated that postulates and common notions (or axioms, in modern terms), which serve as the starting point of geometric proofs, are not synthetic a priori judgments. This does not mean that Poincaré returned to the traditional claim that geometry describes spaces that extend outside and are independent of the human mind. Instead, he stated that although empirical facts can affect the choice of geometric axioms, ultimately they are accepted by virtue of the convention or methodological decision (such a view is called conventionalism). According to this line, we cannot say that Euclidean geometry, indeed any geometric system, is true. The criteria for choosing a given set of axioms are—in addition to avoiding inconsistency—convenience, fruitfulness, and simplicity. We can describe the same physical phenomena by using different geometric systems. According to Poincaré (1905), there are, however, some reasons according to which, "Euclidean geometry is, and will remain, the most convenient" (p. 50):

> 1st, because it is the simplest, and it is not so only because of our mental habits or because of the kind of direct intuition that we have of Euclidean space; it is the simplest in itself, just as a polynomial of the first degree is simpler than a polynomial of the second degree; 2nd, because it sufficiently agrees with the properties of natural solids, those bodies which we can compare and measure by means of our senses. (ibid.)

To sum up, although mathematical intuition understood in Kantian terms played an essential role in Poincaré's philosophy of mathematics, the French scientist claimed that the construction of Euclidean geometry, or at least its starting point, does not emerge from the structure of the transcendental ego, but is instead a matter of convention driven by simplicity and physical facts. Poincaré's views on the foundations of geometry were no exception. Hermann von Helmholtz was another researcher who was strongly influenced by the Königsberg philosopher, but he eventually followed his own path.

Hermann von Helmholtz, who is considered one of the founding fathers of experimental psychology, was undoubtedly one of the most versatile scientists of the nineteenth century. He was a physician, physiologist, and mathematical physicist, but his scientific approach was intertwined with his philosophical interests. His contribution to the understanding of the human mind and nervous system includes, among other things, studies on the perception of space, colors and sounds, the measurement of the speed of nerve impulses, pioneering ideas about unconscious processing, and a critique of nativism. Helmholtz was engaged in mathematical practice, and his interests focused on non-Euclidean geometry (Helmholtz, 1868/1977a). All those multifaceted interests affected his philosophical views on the foundations of geometry (see Biagioli, 2016; Hatfield, 1990; Hyder, 2009).

Helmholtz claimed that Euclidean geometry is not a privileged geometric system. He believed that non-Euclidean geometries are not just "mathematical toys" or useless products of human imagination, but they may be adapted to describe physical reality. According to him, the traditional view that the geometry characterized by Euclid's axioms describes the space that surrounds us is not an irrefutable fact, but rather a question that requires empirical investigation.[7] He also questioned the view—one held even by Poincaré—that Euclidean geometry comes to our minds in a privileged, due to its simplicity, way. In the paper entitled *On the Origin and Significance of the Axioms of Geometry*, Helmholtz (1870/1977b) pointed out that a non-Euclidean world could be imagined as effortlessly as a Euclidean one. By referring to an interpretation of Bolyai-Lobachevsky geometry on a pseudospherical surface—proposed by Eugenio Beltrami—Helmholtz introduced a thought experiment with an imaginary world behind a convex mirror. Biagioli (2016) summarizes Helmholtz's argument as follows:

> for every measurement in our world, there would be a corresponding measurement in the mirror. The hypothetical inhabitant of such a world may not be aware of the contractions of the distances she measures, because these would appear to be contracted only when compared with the results of the corresponding measurements outside the mirror. Therefore, she may adopt Euclidean geometry. At the same time, the geometry of her world would appear to us to be non-Euclidean. (p. 59)

Finally, let us consider Helmholtz's views on the crucial subject of this section, namely geometric intuition. Although he agreed with Kant that sensual impressions are filtered—or transformed—by the perceiver, he rejected the Kantian transcendental standpoint. According to Helmholtz (1870/1977b), "it is no transcendental form of intuition given before all experience" (p. 26). The same applies, according to him, to the ordinary representation of space: it is not permanently fixed—or encoded—in the structure of transcendental ego or any other hardwired factor, but it is formulated through experience, which involves vision and the sense of touch.[8] Therefore, Helmholtz was an empiricist who claimed that the mental capacity traditionally called geometric intuition, "(…) is an empirical acquaintance, obtained by the accumulation and reinforcement in our memory of impressions which recur in the same manner" (ibid., pp. 25–26).[9] Geometric intuition, as well as the internal representation of space, can be analyzed in simpler mental components acquired during individual development. Such an empirical view favored the experimental approach to the development of geometric capacities.

As we have already suggested, Helmholtz's views on the foundations of geometry were influenced by his broader scientific interests. Although we can consider his ideas a prefiguration of the experimental research on the acquisition of geometric competences, we should bear in mind that Helmholtz undertook the problem of the foundations of geometry mainly from the position of a philosopher—thus I have introduced his standpoint in the current section. Before we turn to describing the pioneering psychological research on the development of geometric cognition, let us note that even though the transcendental account of intuition (to a large extent thanks to Helmholtz) found its ultimate place in the history of ideas, Kant's approach has been revived both in philosophical and psychological theories emphasizing innate, or hardwired, components of the human mathematical knowledge.

## 1.4 The development of geometric skills as a psychological problem

Experimental psychology emerged from, or rather began to emancipate itself from, philosophy in the mid-nineteenth century. Hermann von Helmholtz, who interestingly did not describe himself as a psychologist, contributed to the independence of the discipline. The perception of spatial forms, at least in terms of geometric-optical illusions, has been studied since the dawn of experimental psychology (Oppel, 1855; Wundt, 1898). Then, in the first half of the twentieth century, the various flourishing schools of psychology, which were gaining their own methodological maturity at that time, explored geometry-related phenomena. Gestaltists, for instance, aimed to discover the innate principles of perception, thanks to which man is able to perceive spatial patterns and forms as integrated wholes or gestalts (Koffka, 1936), while behaviorists studied how animals learn to navigate in spatial layouts characterized by geometric properties (Tolman, 1932). None of these schools, however, aimed to develop the theory of geometric cognition. The problem was undertaken by Jean Piaget, a Swiss researcher with broad interests, whose innovative experimental methods and pioneering theories on cognitive development extended the scope of child psychology, which had previously focused mainly on sensorimotor development (Piaget, 1926).

In the most general terms, Piaget claimed that to fully attain the cognitive skills observed in adults, the child must pass through several developmental stages, during which subsequent structures of knowledge are constructed—starting in infancy—one after another, and in a fixed order (Flavell, 1963). Although Piaget focused on numerous aspects of cognitive development, from the shaping of naïve physics to social beliefs, his research on the acquisition of knowledge about space was "fairly central to his general theory of intelligence," and thus—as Ninio (1979) continues—it was "developed in great detail" (p. 126). The results of this research were enshrined in two of Piaget's books, namely *The Child's Conception of Space*, coauthored with Bärbel Inhelder (1948/1967), and *The Child's Conception of Geometry*, written together with the latter and Alina Szemińska (1948/1960). The former work, which is of a more introductory nature, focuses mainly on the developmental shifts leading the child to the construction of the concept of Euclidean space, while the later book describes the emergence of specific geometric capacities, such as measuring. In this section, I will introduce and evaluate the content of Piaget and Inhelder's work.

Piaget and Inhelder (1967) claim that the child's mental representation of Euclidean space is not innate (or hardwired), nor is it formed instantly in a mature form. Instead, it emerges, similarly to other cognitive achievements, through the construction process that takes place in a fixed order and overlaps with the general stages of cognitive development, starting in the early period.[10] The construction process is driven not by the passive observation of surroundings but instead by motoric actions, namely exploration of environment and manipulation of objects. These actions play crucial roles, providing the child with opportunities to establish geometric concepts. Such concepts are, however, not abstracted from perceived physical objects, but emerge as the outcome of an internalization of actions. The child—as Piaget and Inhelder (1967) claim—"can only 'abstract' the idea of such a relation as equality on the basis of an action of equalization, the idea of a straight line from the action of following by hand or eye without changing direction, and the idea of an angle from two intersecting movements" (p. 43). Thus, the authors explicitly state that geometric cognition originates in "experimentation," wherein

> the experiments the child performs in modifying objects by his actions are not purely and simply physical experiment dealing exclusively with the intrinsic properties of objects as such (…). The first experiments which give rise to the idea of space are in fact experiments on the subject's own actions, and they consist of finding out how these actions acceded one another. For example, after placing B between A and C, the child discovers that he is bound to encounter it once more between C and A. Having passed the end of a string through a loop, preparatory to making a knot, the child discovers that by pulling it further he does not change the essential character of the knot, and so on. (p. 453)

Let us introduce the order in which, according to Piaget and Inhelder (1967), children develop their geometric skills. Prior to establishing the fact that objects—characterized by the permanence of shape and size—"populate"

Euclidean space, the child constructs so-called *topological space*. According to Piaget and Inhelder, it is characterized by the following topological properties:

- *Proximity* (namely, the "'nearbyness' of elements belonging to the same perceptual field"; ibid., p. 6);
- *Separation* (i.e., two elements are separated when they have no points in common; p. 462);
- *Order* or *spatial succession* (namely, perceiving that "two neighboring though separate elements are ranged one before another"; ibid., p. 7);
- *Enclosure* or *surrounding* (in two-dimensional space, it refers to the situation when "one element may be perceived as surrounded by others," while in a three-dimensional layout, it "takes the form of the relation of 'insaneness,' as in the case of an object in a closed box"; ibid., p. 8);
- *Continuity* (the property that characterizes lines and surfaces; it develops as "the synthesis of" properties listed above; ibid., p. 144).[11]

Although children can discriminate between open and closed visual forms at the earliest developmental stage, they do not have Euclidean concepts—such as angle and length—yet. Thus, according to Piaget and Inhelder, for the child, the earliest form of space is perceptual rather than conceptual. The crucial point is that the topological properties listed above are grasped by children first since they are abstracted from developmentally earliest actions, such as "the dissociated elements of primitive motor rhythms in scribbling" (Clements & Battista, 1992, p. 423).

Piaget and Inhelder's (1967) view that sensitivity to the topological properties listed above precedes the development of the mental representation of Euclidean space is called the topological primacy thesis. The authors grounded their thesis in the results of two kinds of behavioral experiments: haptic and drawing studies (see Ninio, 1979). In the former, the children's task was to explore hidden objects tactilely and match them with replicas. According to Clements and Battista's (1992) summary, "preschool children were reported initially to discriminate objects on the basis of topological features, such as being closed or otherwise topologically equivalent. Only later could they discriminate rectilinear from curvilinear forms and, finally, among rectilinear closed shapes, such as squares and diamonds" (p. 422; see Figure 1.4).

Regarding the latter type of studies, Piaget and Inhelder assumed that since the child's capacity for drawing a copy of a given figure reaches beyond perceptual and motoric capabilities, requiring representational skills, failed, or inexact, copies reflect a deficiency in terms of her mental representation of space. For the first few years of life, the child scrawls but, when she reaches the age of 3, her drawings begin to reflect the topological properties of target objects. The child is, however, unable to understand the difference between curved and straight-sided shapes. For instance, when copying a circle, a drawn line is closed but geometrically

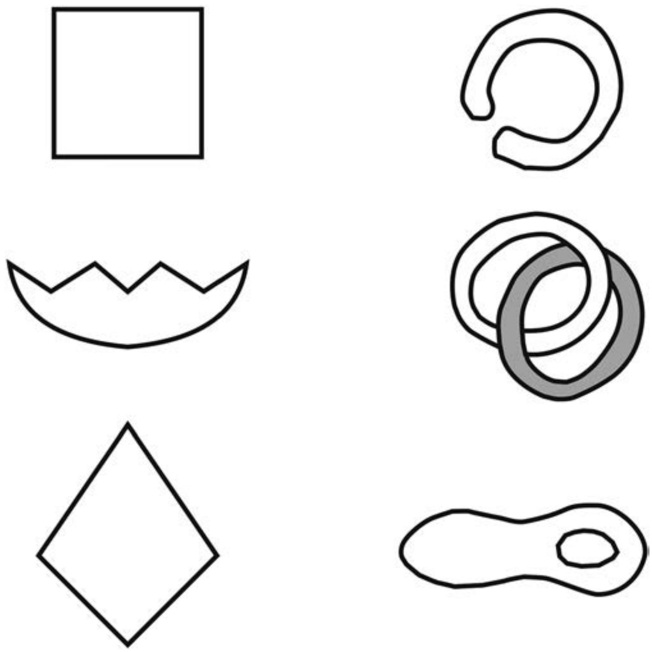

**FIGURE 1.4** Piagetian distinction on Euclidean and topological figures. The figure is patterned after Clemens and Battista (1992, p. 424) and represents stimuli used in behavioral studies by Piaget and Inhelder (1967). According to their topological primacy thesis, children show sensitivity for the figures presented in the right column earlier (in terms of developmental stages). The child begins to understand the figures presented in the left column only in further developmental stages when she becomes sensitive to Euclidean properties. Figures in the left column are called by Piaget and Inhelder "Euclidean," while figures in the right are termed "topological."

irregular. Thus, a circle is indistinguishable for a 3-year-old child from a triangle or rectangle. Approximately at the age of 4, the child distinguishes Euclidean forms substantially better, becoming able to copy the square and rectangle successfully; however, sensitivity for angular properties only refines in the next few years. At the age of 6–7, children reach the sensitivity of all the Euclidean properties, as manifested by the fact that they can replicate the rhombus.

Achieving sensitivity for Euclidean forms and their properties does not mean, however, that children are already equipped with the concept of Euclidean space. Before such a representation becomes fully developed, the child constructs so-called *projective space*. According to Piaget and Inhelder's (1967) terminology, topological properties differ from projective ones in the fact that the former concern a particular object, while the latter concern the object and their relative position to the observer (note that "absolute" distance is not a projective property). As Clements and Battista (1992) note, "projective relations begin psychologically, at the point when the figure is no longer viewed in isolation but begins to be considered in relation to a point of view. For example, the concept of the straight line results from the child's act of taking aim, or sighting. Children perceive a straight line since the earliest years, of course, but they cannot place objects along a straight path not parallel to the edges of a table. Instead, they tend to follow the edges of the table or curve the line toward such a path" (p. 423). This limitation is overcome not earlier than at the age of 7. To sum up, projective space involving the topological one is enriched with a viewpoint. The construction of projective space takes place through linking—or coordinating—possible viewpoints with the planes where the objects are placed.

When the mental construction of projective space is done, children increase their experience of relationships between figures and objects themselves, and begin to grasp the notion that these relationships are organized within a broader frame of reference. Over the following few years, they construct geometric concepts such as a straight line, parallels, and angles and acquire an understanding of metric properties (distance), as well as increasingly recognize the similarity between figures, something that goes hand in hand with increasing transformational abilities. This process leads them, approximately at the age of 12, to developing the concept of *Euclidean space*. Such a concept, in its final form, is highly abstract, because it refers not only to the concrete frame of reference composed of elements occupying currently perceived positions, but rather to the organization of space itself as an invisible "scene" for objects. As Piaget and Inhelder (1967) said:

> It applies equally to positions within the network as to objects occupying any of these positions and enables the relations between them to be maintained invariant, independent of potential displacement of the objects. Thus the frame of reference constitutes a Euclidean space after the fashion of a *container*, relatively independent of the mobile objects *contained* within it, just as projective co-ordination of the totality of potential viewpoints includes each viewpoint actually envisaged (p. 376).

To sum up, Piagetian theory states that the abstract idea of space "populated" by objects and characterized by Euclidean properties is not innate, but is rather a result—or a culminating point—of a long-term developmental process. The process is driven by the progressive coordination of actions taken by the child, and its outcomes do not operate on the principle of all or nothing. Prior to establishing Euclidean space, the child constructs more primitive spaces, namely the perceptual—which is characterized in topological terms—and the projective, which is enriched with possible points of view. During individual development, a new, increasingly sophisticated network of spatial concepts is constructed. These concepts are understood by Piaget and Inhelder (1967) as "(…) internalized actions and not merely mental images of external things or events—or even images of the results of actions" (p. 454). Before we turn to a critical discussion of Piagetian theory, let us consider what geometric intuition—a traditional theme of philosophical investigations—looks like from this perspective. In the general conclusions of *The Child's Conception of Space*, Piaget and Inhelder (1967) refer to this capacity explicitly:

> The "intuition" of space is not a "reading" or apprehension of the properties of objects, but from the very beginning, an action performed on them. It is precisely because it enriches and develops physical reality instead of merely extracting from it a set of ready-made structures, that action is eventually able to transcend physical limitations and create operational schemata which can be formalized and made to function in a purely abstract, deductive fashion. From the rudimentary sensorimotor activity right up to abstract operations, the development of geometrical intuition is that of an activity, in the fullest sense … . (p. 449)

Although the Piagetian theory of spatial development enjoys a certain degree of popularity and remains influential in some academic circles, it has been criticized in various aspects. Most of the doubts raised concern the topological primacy thesis (Darke, 1982), but Piaget and Inhelder's views on the development of projective space (Newcombe, 1989) as well as Euclidean space (Liben, 1978) have also been the subject of criticism (a review of "classic" studies can be found in Clements & Battista, 1992). The simplest objection that comes to mind is the inadequate methodology of drawing studies: the fact that the child cannot draw a copy of a figure correctly does not necessarily mean that she lacks the relevant Euclidean concepts, but may be explained by motor difficulties (Clements & Battista, 1992, p. 423; see also Sinclair, Moss, Hawes, & Stephenson, 2018). Piaget was aware of this objection; therefore, he did not use drawing studies as the only measurement of spatial development (Martin, 1976b). Other objections concern the mathematical adequacy of terms employed by Piaget and Inhelder (1967) and are related to the division of figures into "topological" and "Euclidean" (Clements and Battista, 1992, p. 424ff). Given that experimental design—as well as each observation—is theory-laden, such objections permit us to doubt the reliability of the Piagetian assertion that Euclidean properties develop from topological ones.

Let us start with clarifying what the topology is. In the simplest terms, it is a branch of mathematics (precisely, geometry) that investigates the properties of space conserved under the class of continuous deformations. This class includes operations such as stretching, twisting, crushing, and bending, but excludes gluing and tearing. *Proximity* is the first of the relationships described by Piaget and Inhelder (1967) as topological. Recall that the authors define it as "'nearbyness' of elements belonging to the same perceptual field" (p. 6). According to Kapadia (1974), "this is certainly not a topological relationship. For it involves a vague idea of distance, a concept foreign to general topology: there is no difference, topologically, between a man wearing a pair of shoes and the same man who has merely taken off his shoes" (p. 420). Also, *enclosure* cannot be perceived—even though Piaget and Inhelder do this—as a topological property. At this point it should also be noted that other notions used in *The Child's Conception of Space* do not correspond to standard mathematical usage (i.e., *separation*), or they are not well defined (i.e., *continuity*) (Martin, 1976b).

These issues are related to the mutually exclusive division of figures into "topological" and "Euclidean" employed by Piaget and Inhelder in their experimental designs. As Martin pointed out, they, however, did not specify the criterion of this division. Anyhow, as the author noted, such a classification cannot be maintained since each figure has both topological and Euclidean properties to the same degree: "for any one figure has as many topological properties as any other figure, and the same is true for Euclidean properties" (Martin, 1976b, p. 10). According to this interpretation, some figures used as experimental stimuli were in fact topologically equivalent and thus Piaget and Inhelder's reasoning that child performance depends on topological properties is unsound. After discussing some of the conceptual doubts and their impact on experimental designs, let us look at the results of the direct replications of Piagetian experiments and other findings that contribute to the evaluation of the topological primacy thesis.

The attempt to directly replicate Piaget and Inhelder's original findings turned out to be a severe challenge due to the short, or even skimpy, descriptions of the testing conditions and the suspicion that these conditions varied among all the tested children (Page, 1959). However, this does not change the fact that replication studies, whether they are of drawing or haptic experiments, were already carried out a few years after the release of the first English-language version of *The Child's Conception of Space* (Lovell, 1959; Page, 1959; Peel, 1959). These studies generally replicated the original findings, and simultaneously corroborated the topological primacy thesis, while at the same time revealing some anomalies. On the one hand, "the children between 2–5 and 4–0 years were nearly three times as successful with the topological forms as with the simple geometrical shapes" (Page, 1959, p. 119). On the other hand, Page's (1959) study challenged Piaget and Inhelder's remarks on the poor distinguishing of curvilinear and rectilinear forms by 4-year-old children: "the children tested in the present experiment had much more competence in this matter than these remarks might lead one to expect" (p. 117). Regardless of this, we can suppose that the results may be task dependent: due to doubts about the theoretical basis of Piaget's experimental designs, and insufficient descriptions of the target experiments, the reliability of Page's study (as well as other direct replications of the original findings) is under question.

Subsequent research, reaching beyond direct replication, revealed further effects incoherent with Piaget and Inhelder's claims. For instance, Martin (1976a) tested whether a child's representation of space requires the preservation of the topological properties of forms when those forms are transformed in various ways. To this end, he used six target-shapes and their transformations of three kinds. Transformed items of the first kind were topological counterparts of the target, whereas transformed items of other types aimed at eliminating a specific topological relationship, namely connectedness (the second kind) or closedness (the third kind), while simultaneously maintaining Euclidean characteristics (namely, straightness, curvature, length of a line segment, or angle) as much as possible.[12] Martin tested 90 children, with 30 each from ages 4, 6, and 8. In each trial, the child was first confronted with the target-shape, and then with three transformed items. Finally, the child was asked to indicate two items: the one "most like" the target-shape, as well as the "worst" modification. The study revealed that although children at the age of 4 tended to indicate topologically equivalent transformations as "the worst" counterparts of the target-shapes less frequently than 6- and 8-year-olds, "the worst" scores turned out to be at or above chance. Furthermore, the youngest children selected transformations that did not preserve topological properties as "most like" the target-shape with similar frequency as older participants. The results, as suggested by Martin (1976a), "do not support the theory that topological concepts develop prior to Euclidean and projective concepts in the child's representational space" (p. 37).

Let us summarize the objections toward Piaget and Inhelder's theory regarding the development of spatial cognition and add some new elements. Both methodological considerations and the results of experiments, for example, by Martin (1976a), suggest that the theory fails in regards to the order of emerging spatial concepts. Notably, the crucial claim, namely that grasping topological properties precedes the acquisition of projective and Euclidean concepts, turns out to be difficult to maintain. As I have mentioned earlier, Piaget and Inhelder's results seem to be task dependent, or as Clements and Battista (1992) noted, they may be just artifacts emerging from selected visual stimuli and the children's sensitivity to them. Such a capacity, as these authors claim, may not be a matter of grasping topological properties by the youngest children first, but rather a derivative of familiarity with some figures or their perceptually salient properties. As Clements and Battista (1992) continue, "It may be that children do not construct first topological and later projective and Euclidean ideas. Rather, it may be that ideas of all types develop over time, becoming increasingly integrated and synthesized" (pp. 425–426).

Despite all of this, we should admit that Piaget and Inhelder's studies on child development of geometric concepts were precursory. Their value—as the first genuinely experimental approach to the problem—is unquestionable. The contribution of these researchers to understanding geometric cognition is also pioneering in another manner. In a similar manner to Helmholtz, Piaget and Inhelder rejected the innateness of geometric competences, but simultaneously shifted the received view that Euclidean concepts emerge from the perception of the surrounding world. Instead, they emphasized the role of the exploration of the environment, the manipulation of objects, and the internalization of these activities in the construction of geometric intuitions. As we will see in Chapter 2, the contemporary understanding of the cognitive foundations of geometry dismisses the constructivism of Piaget and Inhelder, showing that some Euclidean concepts are "hardwired" in the child's mind, while agreeing with the Swiss researchers that reciprocal relationships between perceptual and motoric activity and the internalization of one's own actions are crucial for refining geometric skills. Before we do this—and which will be preceded by looking at the emergence of the cognitive science of mathematics—let us introduce an educational perspective on the development of geometric skills.

## 1.5 Euclid in the classroom

One of the milestones of Piaget's work was making cognitive development the subject of experimental research. To a large extent, developmental psychology reached beyond the simple description of changes in a child's behavior over time thanks to his contribution. The Piagetian approach went hand in hand with other disciplines for which "development" is a crucial term. It inspired especially educational studies, which have applied both constructivism and the idea of developmental stages (Egan, 1983). The problem of learning mathematics is not an exception: several approaches state that mathematical knowledge is acquired by means of a construction process, characterized by distinct stages achieved by the student in a fixed order (Kamii & Ewing, 1996; Lerman, 1989). Regarding the field of geometry, a wife-and-husband duo of Dutch scholars, Dina van Hiele-Geldof and Pierre van Hiele (1957/1984), elaborated in their doctoral dissertations a prominent theoretical model (see also van Hiele, 1986), which influenced both educational psychology research and mathematical curricula of several countries, including the United States (see Battista, 2007; Clements & Battista, 1992; Usiskin, 1982; Wirszup, 1976). The model consists of three main component-parts: a description of theoretical assumptions, levels of geometric thinking, and, finally, phases of instruction. In the current section, I will first introduce the model, and then discuss it in relation to the results of educational psychology studies.

The starting point of the van Hieles' model is a set of theoretical assumptions that naturally bring to mind Piaget's ideas. But, as we will see later, thinking of the van Hieles' model as embedding geometric education in the received framework would be wrong. First of all, van Hieles propose that learning geometry is a sequential and hierarchical process (see Battista, 2007; Clements & Battista, 1992; Hershkowitz, 2009; Roth, 2011). It is sequential since achieving the mastery of academic geometry involves passing through the several levels, each characterized by different modes of thinking. The process is also hierarchical since mastering each of the higher levels requires proficiency at a lower one; in other words, the student cannot skip a lower level and reach a higher one. Another assumption made by van Hieles—linked, however, with the previous one—is that the explicit understanding of geometric concepts at some level is preceded by the implicit grasping of their content at the previous one. Finally, the researchers assume that geometric concepts on each level are structured by linguistic symbols that are unique at the appropriate level. According to these, for instance, the correct understanding of the relationship between a square and a rectangle at some level may be incorrect at another one, which can be reflected in communication problems between the teacher and the student. However, it is not that the language merely reflects the development of geometric concepts. Instead, as Clements and Battista (1992) note, "language structure is a critical factor in the movement through the levels" (p. 427; see Chapter XIII of Dina van Hiele-Geldof's doctoral dissertation accessible in van Hiele & van Hiele-Geldof, 1984). Now let us introduce the levels of the development of geometric thinking.[13]

The first level is called the "visual." At the beginning of education, the student grasps geometric figures as gestalts relying only on the purely visual aspects, and she does not consider their component parts, geometric properties, or the fact that they belong to the more general geometric category. The recognition of the figure has a holistic character and leads the student to create her own mental image. At this level, the student learns to use verbal labels of particular figures, wherein if she calls the observed figure a rectangle, she means in fact, that the figure's overall shape "fits" the shape called (by the instructor) "rectangle." The performance in tasks with distinguishing figures or recognizing their congruency depends solely on holistic visual properties ("these figures appear to be the same") and does not involve the consideration of individual properties ("these figures are the same because they have four angles"). At this level, geometric forms are categorized only through their similarity with familiar visual prototypes: for example, the student who is asked, "Why is this a rectangle?" and answers something like, "Because it reminds me of a door," cannot justify her statement by referring to geometric relationships. Only after the transition to the second level does the student begin to understand that visual objects can be classified into more general categories and that these objects may be characterized by specific properties.

At the second level—known as "descriptive" or "analytic"—the student is able not only to identify figures by their overall visual shape, but also to characterize figures due to the properties associated with them. At this level, the student grasps that a rectangle is a parallelogram containing a right angle, and therefore applies the verbal label "rectangle" to objects characterized by properties that she has learned to specify as "rectangles." Even though adepts of geometry at the descriptive/analytic level still perceive figures holistically, now they are no longer just visual gestalts, but bundles of relationships. These relationships are grasped not only through passive observation but also by actions such as drawing or measuring. As Clements and Battista (1992) note, at the second level "students discover that some combinations of properties signal a class of figures and some do not; thus, the seeds of geometric implication are planted" (p. 427). Nevertheless, geometric thinking at this level is still limited, since the student does not yet correctly understand how the properties of particular geometric forms are related to each other. This limitation is overcome with the transition to the following level.

The third level is called the "abstract" or "relational." Once the student reaches this level, she not only knows the properties of a given figure but also understands hierarchical relationships between geometric forms and their properties. This capacity allows her to categorize figures and informally substantiate these categorizations (a figure resembles another one; however, it is not identical with it since it has some additional characteristics). Furthermore, at this level, students reach beyond identifying figures based on visual shape and some properties, becoming capable of understanding geometric concepts in terms of necessary and sufficient conditions. This shift enables them to understand and demonstrate logically sound geometric reasoning. Plato's demonstration introduced in Section 1.3 seems to be a good example of the reasoning accessible on the third of the van Hieles' levels. To demonstrate that a square built on the diagonal of a given square is double, the student should be aware of some geometric relationships, for instance, the fact that each square can be divided into two triangles, wherein a diagonal of a square constitutes a hypotenuse (the side opposite the right angle) of these triangles. Nota bene, as we remember, the demonstration enshrined in the *Meno* is carried out by the geometrically naïve slave boy, which is—according to Plato—an argument in favor of the innateness of geometric intuition. In contrast to the philosopher, the van Hieles' model states that the capacity to conduct informal reasonings appears at a relatively late stage of geometric education. Despite the fact that students at the third level understand that a definition is a tool of the logical organization of geometric relationships (therefore this level is called "relational"), and geometric forms about which they conduct informal reasonings are not just concrete visual forms but bundles of unchanging properties (thus the level is also called "abstract"), they still do not know that formal deduction serves as the tool for proving geometric theorems.

At "the level of formal deduction," which is the fourth according to the numbering adopted here, "thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions" (van Hiele & van Hiele-Geldof, 1984, p. 246). The student is familiar with the axiomatic-deductive method and can use it in practice to demonstrate some geometric truths via the construction of formal proofs. Having achieved this level, she understands notions like "definition," "axiom," "theory," and "proof," and efficiently distinguishes between defined and undefined terms. The subject of reasoning carried out by students at the fourth level involves the relations of the properties of categories of geometric forms. Adepts of geometry also achieve mastery in understanding so-called second-order relationships (namely involving relationships of relationships) considered within a formal system. Although geometric knowledge at this stage is already highly professionalized, there is still a final level that can be achieved.

After the transition to the fifth level, called "meta-mathematical," the geometer's (because at this level it would be strange to talk about a student, at least in the ordinary sense) capacities reach beyond formal reasoning about geometric relationships *within* the Euclidean system, and begin to concern multiple, that is, non-Euclidean, systems. Geometers, as Clements and Battista (1992) note, "can study geometry in the absence of reference models, and they can reason by formally manipulating geometric statements such as axioms, definitions, and theorems. The objects of this reasoning are relationships between formal constructs. The product of their reasoning is the establishment, elaboration, and comparison of axiomatic systems of geometry" (p. 428). According to the van Hieles' model, the transition to the meta-mathematical level means achieving the highest proficiency in geometric thinking by the adept.

Since we already know the characteristics of the levels, let us briefly introduce phases of instruction that allow the student to become skilled at every level of geometric thinking. First of all, the term "instruction" should be understood literally, because passing through the levels listed above is not a matter of chronological age. Instead, progress in geometric thinking, at least on the first three levels, strictly depends on the learning process that is facilitated by the teacher and the school curriculum. In the first, the so-called "information," phase, students become familiar with the scope of the field through discussions with the teacher. "Guided orientation" is the second phase. At this stage, the student is involved in the active exploration and manipulation of objects and the teacher guides her to become implicitly familiar with selected geometric concepts and methods. During the third, "explicitation," phase, students begin to understand geometric concepts explicitly, which manifests itself in their appropriate linguistic descriptions of the topic. When students already practice forming such reports in their own words, the teacher familiarizes them bit by bit with relevant portions of professional geometric language. Thus, with the help of instruction, students' descriptions of geometric matter become less arbitrary. The fourth phase is called "free orientation," since the student can apply the acquired conceptual knowledge and portions of professional terminology to solve problems independently. The role of the teacher in this phase is primarily correcting students' mistakes and introducing alternative ways of solving the problem. Finally, in the fifth phase ("integration"), the student integrates the acquired knowledge and skills into a coherent framework, which may be relatively easily reported in the professional language of geometry. The teacher fills the gaps (if any) in her knowledge and indicates mutual relationships between the elements of the subject matter. If this task is completed, the student is ready to transition to the next level of geometric thinking.

As I have already mentioned, the van Hieles' model has been widely recognized by educators and has become the theoretical basis of curricula in many countries. On the other hand, further studies raised doubts regarding the adequacy of the initial assumptions, number of theoretical levels, and relationship between geometric development and general cognitive development, as well as the educational attainment of students in reference to levels (see Roth, 2011). Before we present the controversies, let us note that the results of numerous studies turned out to be, at least partially, consistent with the model, and validated its potential usefulness in describing geometric development (see Clements & Battista, 1992 and Battista, 2007, p. 428). For instance, Burger and Shaughnessy (1986) conducted a study with the participation of students from the first grade up until college and asked them to perform tasks derived from the characteristics of the van Hieles' levels. The tasks mainly involved drawing, identifying, and categorizing shapes, and informal as well as formal problem-solving in the domain of geometry.

Burger and Shaughnessy (1986) found that the youngest participants revealed a tendency to identify forms by means of their visual prototypes ("a rectangle reminds me of a door") and to characterize them in reference to geometrically irrelevant properties. These students were included in Level 1 of geometric thinking. In turn, more advanced participants who operated on shapes based on their properties ("a rectangle has two sides equal and parallel to each other") were counted as attaining Level 2. Students who demonstrated the correct understanding of relationships between different shapes ("both a rhombus and a rectangle are parallelograms, just as a square") were included in Level 3 of geometric thinking. Finally, only one student who showed the capacity to conduct formal proofs was regarded as reaching Level 4. The study does not provide information about Level 5, which is not surprising since highly advanced meta-mathematical considerations reach beyond school, or even college, material and only seem to be available at the stage of doctoral studies in logic or mathematics.

Burger and Shaughnessy also found that the participants assigned to particular levels used specific linguistic expressions that substantiated one of the crucial assumptions of the model. On the other hand, another critical one, namely the discreteness of levels, previously found as accurate (see Wirszup, 1976), was not confirmed by the results. As Burger and Shaughnessy (1986) note "the levels appear to be dynamic rather than static and of a more continuous nature than their discrete descriptions would lead one to believe. Students may move back and forth between levels quite a few times while they are in transition from one level to the next" (p. 45). Perceiving the levels as dynamic and continuous is supported by the finding that the student can be assigned simultaneously to different levels depending on the task: she solves, for instance, some problems in an abstract/relational way (Level 3), while others rather by employing a less advanced descriptive/analytic mode of geometric thinking (Level 2). Some other studies, for example, with the participation of undergraduate teachers, supported this line (Mayberry, 1983). Furthermore, Battista (2007) challenged the assumption that different kinds of reasoning characterizing levels develop sequentially. According to him, it is possible that "visual-holistic knowledge, descriptive verbal knowledge, and, to a lesser extent initially, abstract symbolic knowledge grows simultaneously, as do interconnections between levels" (p. 850). This, however, does not exclude the fact, as Battista continues, that despite the simultaneous development, "one level tends to become ascendant or privileged in a child's orientation toward geometric problems" (ibid.).

Another challenge concerns the base level. As we already know, assigning the student to a given level depends on fulfilling the level's characteristics manifested in her behavior. But what if she does not meet all of the indicators of the first level? Fuys, Geddes, and Tischler (1988) suggested characterizing geometric thinking in such cases as "the weak visual level." Clements and Battista (1992) went a step further, postulating "the existence of thinking more primitive than, and probably prerequisite to, van Hiele's Level 1" (p. 429). The zero level, called "pre-recognition," involves only processing a part of a shape's attributes by the child (we cannot talk about a student here, since the authors place this level at the earliest Piagetian stage of spatial development). At the level of pre-recognition, the child recognizes the difference between curvilinear (e.g., circle) and rectilinear forms (rectangle); she is not, however, capable of discriminating forms belonging to the same category.

At the end of this section, let us look at geometric education from a more practical perspective, namely in terms of student achievements. The results of a Programme for International Student Assessment (PISA) study by the Organisation for Economic Co-operation and Development (see OECD, 2014) revealed that students from several Western countries were less proficient in geometry (to be clear, the relevant subscale is called Space and Shape) in juxtaposition to other fields of mathematical education. The United States and United Kingdom were the countries where the divergence was the largest. According to Mammarella, Giofre, and Caviola (2017), a potential reason lies in the fact that only 8%–15% of the content of mathematical textbooks used in American primary schools—which reflects the scope of the curricula that they are intended to deliver—concerns geometry.

Furthermore, given that progress in geometric thinking depends on the instruction received, the authors see the second reason for students' geometric difficulties in the fact that teachers in the United Kingdom and United States generally reach only the first and the second of the van Hieles' levels (see Clements & Sarama, 2011). Last, but not least, Mammarella and colleagues (referring to their own research on the Italian population) suggest that if the assumption of the van Hieles' model regarding sequentiality of geometric development is correct, one can suspect that difficulties at the earliest levels (visual and descriptive/analytic) can meaningfully hinder progress in higher geometric thinking (abstract/relational, formal deduction). Therefore, the authors emphasize the need to facilitate the geometric thinking of both teachers and students through already designed and future psycho-educational interventions (see Mammarella et al., 2017, pp. 237–240). Taking into account the fact that curricula and interventions should be evidence based (S. P. Miller & Hudson, 2007), and that the "cognitive revolution," which started in the mid-1950s, is by no means over yet, mathematical educators should constantly follow the development of the cognitive science of mathematics. In the following section, we will look at the growth and development of this research field and discuss the place of geometric cognition within it.

## 1.6 How cognitive science discovered and forgot Euclid

Cognition, understood as an activity of the human mind that involves forming internal representations including concepts, and processing them to guide actions, has been studied experimentally at least since the time of Piaget. The 1920s–1950s works by the Swiss psychologist were not, however, widely known in the United States,[14] where behaviorism—intentionally avoiding speculations on internal mental states—flourished as the dominant research perspective. The situation in the United States changed in the mid-1950s, when the so-called "cognitive revolution" began (see Bechtel, Abrahamsen, & G. Graham, 1998; Gardner, 1985; G. A. Miller, 2003). Although the name was created and accepted by the academic milieu a few years later, September 11, 1956, is assumed as the symbolic date of birth of cognitive science. On that day, a famous symposium, organized by the Special Interest Group in Information Theory, took place at the Massachusetts Institute of Technology. Among others, Newell and Simon talked about their computer program (programmed together with Shaw) that was capable of solving mathematical problems, Chomsky introduced the influential idea of transformational generative grammar, and G. A. Miller discussed the results of his research on the limits of short-term memory.

After many years, the latter of these researchers, G. A. Miller (2003), noted: "I left the symposium with a conviction, more intuitive than rational, that experimental psychology, theoretical linguistics, and the computer simulation of cognitive processes were all pieces from a larger whole and that the future would see a progressive elaboration and coordination of their shared concerns" (p. 143).[15] In addition to experimental psychology, linguistics and computer science, neuroscience, philosophy, and also anthropology were constituents of cognitive science from the very beginning; however, the position of the former trio was perceived as central, while the latter were seen as more peripheral. This state changed in the 1980s, with the search for the unity and integrity of this multibranch discipline being done in earnest for the first time (Miłkowski, 2016; 2017; see however, Núñez et al., 2019). Cognitive science is, however, still an open enterprise that incorporates the results of related research fields if necessary. Regarding the latter, and as we shall see in Chapter 2, the results of ethology, comparative psychology, and evolutionary biology appear to be especially informative. At this point we should, however, note that the early cognitive science studies on mathematical thinking were "less interdisciplinary," being based mainly on the methods of computational modeling and experimental psychology. Prior to the introduction of the first cognitive studies on geometry, let us also add that early cognitive science conceptualized mental processing mainly in terms of transformations of amodal and arbitrary symbols (Fodor, 1975; Jackendoff, 2002), performed by brain structures distinct from these engaged in sensorimotor processing (Bechtel et al., 1998).

Mathematics, especially mathematical logic, was an important research topic for this budding discipline. Some of the "founding fathers" of cognitive science had directly recognized that the solving of logico-mathematical problems was the prototypical matter of human thinking. For instance, Newell and Simon's (1956) aforementioned *artificial intelligence* program (nota bene, this term was also coined in 1956), known as The Logic Theorist, attempted to prove some theorems enshrined in Russell and Whitehead's *Principia Mathematica*. Interestingly, a proof of one of the theorems (i.e., 2.85) done by the computer program was recognized as more elegant than the original. A few years later, Newell, Shaw, and Simon (1959) finished their work on a new AI program called General Problem Solver. Although the name is somewhat exaggerated, since the class of potentially solved problems was limited solely to sufficiently formalized ones, the General Problem Solver and similar programs performed well in various kinds of mathematical problems. Euclidean proofs are not the exception here, yet before introducing geometric theorem-proving programs it is worth casting an eye over the basic designing principles of early AI programs.

General research methodology and conceptualization of cognition are more important than technical details. First of all, Newell and Simon (1972; 1976), as well as some of the other "founding fathers" of cognitive science, made efforts to make the computational models psychologically reliable. To this end, they implemented the results of psychological research into programs, mainly verbal protocols, on problem-solving by flesh-and-blood human beings (Ericsson & Simon, 1984). It should be emphasized that although these data served as empirical support of the theoretical proposal embodied within the program, the computer simulations delivered predictions that could be empirically tested in further studies. Regarding the conceptualization of thinking (or cognition), Newell and Simon's (1976) studies were deeply embedded in the already well-known idea that cognitive processing is based on manipulating amodal and abstract symbols. In particular, these researchers proposed the so-called physical symbol system hypothesis, which states that "a physical symbol system has the necessary and sufficient means for general intelligent action" (p. 116). More specifically, Newell and Simon understood thinking of each physical symbol system, including both human and AI programs, as the heuristic search for a solution to a problem carried out as rule-based and sequential (i.e., step by step) symbol manipulations. Such an approach was also applied in early cognitive studies on geometry.

The first—to the best of my knowledge—computer program devoted to geometric theorem-proving was designed in the early 1960s by Gelernter (1963). While the program did not implement results on human problem-solving research, its abilities were nevertheless impressive for those times. The program, for instance, was capable of finding proofs for propositions such as "a point on the bisector of an angle is equidistant from the sides of the angle" and "in a quadrilateral with one pair of opposite sides equal and parallel, the other pair of sides are equal" (ibid., p. 143ff; see Figure 1.5).

Regarding the form of representation, Gelernter's program operated by transforming symbols according to syntactic rules, but also directly by processing

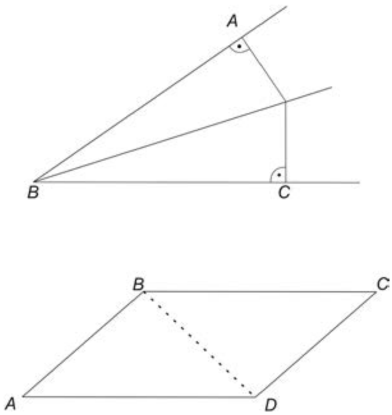


**FIGURE 1.5** Diagrams for the propositions proved by Gelernter's program. The figure is patterned after Gelernter (1963, pp. 147–148) and depicts diagrams for the two propositions proven by his geometric theorem-proving program. The propositions are the following: "a point on the bisector of an angle is equidistant from the sides of the angle" (top) and "in a quadrilateral with one pair of opposite sides equal and parallel, the other pair of sides are equal" (bottom).

diagrammatic representations of the relevant figures. Thus, an executive routine of Gelernter's program (called a heuristic computer) involved two main component parts, namely, a syntax computer and a diagram computer (see Figure 1.6). Gelernter supposed that such a computational organization reflects human geometric practices. As Simon (1978) comments, "before the system attempted to prove syntactically that corresponding angles, say, or corresponding sides of a pair of triangles were equal, it first tested for approximate equality on the diagram. The space of the diagram therefore served as a planning space that prescreened proof attempts and saved effort in fruitless proof attempts" (p. 15). The path delineated by Gelernter was followed successively by other researchers.

Nevins (1975) designed another theorem-proving artificial system that was able to solve a wide class of plane geometric problems. Although the processing

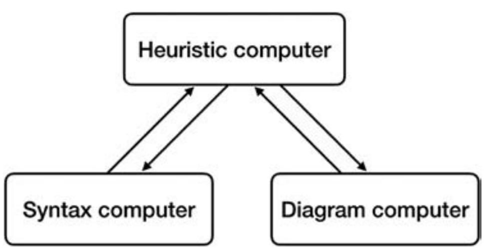


**FIGURE 1.6** The structure of Gelernter's program. The figure is based on Gelernter (1963, p. 139) and shows the main component parts and the information flow of his geometric theorem-proving program.

of curve lines or the introduction of new points into diagrams was beyond its powers, the author included the capacity to process several geometric predicates into the program, such as straight line, parallel, right angle, equal segment, or equal angle, which allowed it to operate in an abstract problem space in an effective way. The Nevins program was an important step toward the powerful automation of geometric reasoning, however its explanatory value was limited since the operational rules were not based on human performance data. This limitation was overcome by Greeno (1978), who used students' verbal reports to design a geometry theorem-proving program called Perdix. These reports revealed, for example, that geometric reasoning is largely based on the processing of diagrammatic information. Notably, in contrast to the models of Gelernter and Nevins, Perdix's purpose was not to solve geometric problems in the most effective way, but rather by modeling the performance of students with moderate expertise in geometry. A similar motivation guided the Geometry Tutor Expert (GTE) by J. R. Anderson, Boyle, and Yost (1985). The GTE was designed to be highly psychologically reliable. To this end, the authors implemented a number of solutions adopted previously in a famous cognitive architecture called Adaptive Control of Thought* (ACT*) by J. R. Anderson (1983). In particular, the researchers incorporated ACT*'s heuristics of predicting inferences on the basis of acquired contextual features involving both the diagram's properties and justified statements.

At the end of this brief review of classical AI geometric programs, let us introduce Koedinger and J. R. Anderson (1990) Diagram Configuration (DC) model. The model was well grounded in verbal protocols, and its purpose was to capture the kind of geometric reasoning made by experts. The latter turned out to solve geometric problems by sketching out possible solutions and planning the following inferences by taking into account information contained in the diagram. This result is consistent with both previous computer simulations (Greeno, 1978), as well as grounded in notable theoretical claims by Larkin and Simon (1987). In the meaningfully entitled article, *Why a Diagram Is (Sometimes) Worth Ten Thousand Words*, Larkin and Simon noticed that a diagram predominantly represents related information in a more compact way than a set of statements (where the information may be fragmented), and thus facilitates the process of understanding. Furthermore, according to Larkin and Simon, a diagram supports perceptual inferences that can be equally robust, but easier to perform. In contrast to previous computer programs, which initially planned successive inferences in a step-by-step way, but in line of verbal protocols obtained from human experts, DC was capable of step-skipping and purposive planning inferences through "parsing geometry problem diagrams into perceptual chunks, called diagram configurations, which cue relevant knowledge" (Koedinger & J. R. Anderson, 1990, p. 511).

As we have seen, geometry was the research topic of the "computational branch" of cognitive science from the outset. Although research on geometric theorem-proving computer programs was not completely isolated from the other empirical disciplines constituting cognitive science,[16] we can venture to say that in the first decades psychology, neuroscience, linguistics, and anthropology did not contribute to exploring geometric cognition from their own perspective. The situation was different if we consider the beginnings of psychological research on numerical cognition. Already in the 1960s, Moyer and Landauer (1967) published the results of a behavioral study which measured reaction time (RT), and which strongly contributed to the further exploring of number processing. They discovered the so-called numerical distance effect: when participants are asked to select the numerically larger value of two presented digits, RT increases when the numerical distance between them decreases. Moyer and Landauer also observed the so-called numerical size effect: for the same numerical distance, RT is shorter when participants compare small numbers than larger. These effects have been intensively studied in further behavioral experiments (see e.g., Dehaene, 1989; Tzelgov, Meyer, & Henik, 1992), and then, when the neuroimaging techniques have been widely available, also in neurocognitive ones (see e.g., Cohen-Kadosh, Lammertyn, & Izard, 2008; G. Wood, Nuerk, & Willmes, 2006).

Although I have no intention to present the history of studies on numerical cognition, I have introduced the example of Moyer and Landauer's study since it illustrates the quite different beginnings of the investigation of two domains of mathematical thinking. In the case of numerical cognition, early efforts—driven primarily by psychological methods—were "tuned" to discovering the elementary numerical processing by each person familiar with numbers (deciding which number is greater does not require specialist knowledge). On the other hand, the earliest computational studies on geometric cognition instantly aimed to understand the manifestations of higher geometric thinking, such as proving theorems, that require extensive training, but "jumped over" elementary geometric cognition. Research on the latter, as we will see in Chapter 2, began only in the mid-1980s. Admittedly, numerous behavioral studies using geometric stimuli had already been conducted in the 1970s. For instance, Shepard and colleagues conducted a well-known study on mental imagery and found that in a matching task requiring one to decide whether a geometric object is the same but rotated or a mirror of a target, participants' RTs increase with the angle of rotation (L. A. Cooper & Shepard, 1973; Shepard & Metzler, 1971). Considering the contribution of the result for understanding geometric cognition, we should, however, bear in mind that the researchers were interested prima facie in the format of mental representations, but not in angle processing in general. Let me put these issues temporarily aside and consider how the development of cognitive science affected the studies on mathematical thinking.

As we remember, cognitive science was dominated from the outset by the triumvirate of computer science, psychology, and theoretical linguistics (in practice, Chomsky's generative theory did not influence early studies on mathematical thinking), while neuroscience, anthropology, and philosophy played a peripheral role. During the 1980s, the situation began to change. As Bechtel and colleagues (1998) nicely summarized the situation, cognitive science "regained that breadth and more by expanding in two directions: vertically into the brain and horizontally into the environment" (p. 81). At that time, extensive research in all cognitive science branches was carried out, but embodied cognition—a new grand perspective on the nature of cognitive processes—also came to the fore.[17] Although there are numerous ways of understanding what "embodiment of the mind" really is, most researchers agree that it refers to claims that cognitive processes are causally grounded in sensorimotor activity and that the body shapes (constrains, enables, or even constitutes) the mental activity (see e.g., Barsalou, 1999; Chemero, 2011; Clark, 1998; Davis & Markman, 2012; Lakoff & Johnson, 1980; M. Wilson, 2002).

Besides the fact that embodied cognition has been widely applied in all branches of cognitive science—starting from experimental psychology (Glenberg, 2010), through linguistics (Lakoff, 1987), philosophy (Varela, Thompson, & Rosch, 1991), and anthropology (Csordas, 1990), and ending at computer science with its new faces, such as behavioral robotics (M. L. Anderson, 2003)—this perspective has strongly influenced studies in interdisciplinary research fields, such as social mind (Brożek, 2013; Niedenthal, Winkielman, Mondillon, & Vermeulen, 2009), aesthetics (Fingerhut & Prinz, 2018; Matyja, 2016), or psychopathology (Fuchs & Schlimme, 2009). Mathematical thinking was not an exception there. Let us start introducing the embodied mathematics by recalling that Lakoff and Johnson (1980) proposed that the entire conceptual system emerges from sensorimotor activity. Concrete concepts grow directly from perceptual and motor experiences, accumulated as so-called image-schemas (see Johnson, 2012), while abstract ones are produced by metaphorical mappings. According to these authors, a metaphor is not just a linguistic expression, but rather "a cognitive tool," reflected in language use and built by two conceptual domains, namely "the target domain, which is constituted by the immediate subject matter, and the source domain, in which important metaphorical reasoning takes place and that provides the source concepts used in that reasoning" (Lakoff & Johnson, 1980, p. 185). Furthermore, metaphorical expressions such as "I demolished his argument," "you're wasting my time," or "our love relationship is at a crossroads," are meaningful for us since the abstract concepts "argument," "time," or "love" preserve the inference structure of concrete concepts, respectively, "war," "money," and "journey," which directly arise from bodily experience.

A few years later, George Lakoff (1987, pp. 353–369), inspired by the speculative ideas of the cofounder of mathematical category theory, Saunders Mac Lane (1986), proposed that logical and mathematical concepts are also deeply rooted in sensorimotor activity via metaphorical mappings. For instance, Lakoff has noted that an idea of "class" (or "set") arises from an ordinary concept of "container," and a "subclass" originates in bodily experience with part-whole. A decade later, Lakoff approached mathematical cognition again in cooperation with Rafael Núñez, proposing the metaphorical grounding of various concepts, from fields as numbers (e.g., Numbers Are Points on the Line), arithmetic (e.g., Arithmetic Is Motion Along the Path), set theory (e.g., Sets Are Objects), or functions (e.g., The Domain of the Function Is a Collection of Acceptable Input Objects) (Lakoff & Núñez, 1997; Núñez & Lakoff, 1998). These works were only a prelude for Lakoff and Núñez's (2000) book entitled *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being*, which unified the perspectives of cognitive psychology, linguistics, and philosophy of mathematics to substantiate the claim that "the detailed nature of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason" (p. 5). On the pages of this book, the authors developed and presented their previous studies in more detail, but also enriched embodied mathematics with new far-reaching hypotheses (e.g., regarding the bodily roots of both potential and actual infinity) and case studies, e.g., the cognitive structure of Euler's famous identity:

$$e^{\pi i} + 1 = 0.$$

In the meantime, somewhere between Lakoff's initial idea about the possibility of applying the category of embodiment to studying mathematical cognition and the publication of *Where Mathematics Comes From*, numerous experimental studies on the cognitive processing of numbers by human (both adults and infants) and nonhuman animals were conducted. Furthermore, the classic paradigms of measuring various aspects of this processing, both in a purely behavioral and a neuroscientific manner, were established (Berch, Geary, & Koepke, 2016; Campbell, 2005; Cohen Kadosh & Dowker, 2015; Dehaene, 2011; Dehaene & Brannon, 2011; Geary, Berch, & Koepke, 2015; given the sizeable nature of the literature, I list only some of the "classic" handbooks and review collections here). Some of them were used to test the assumptions of the embodiment of mathematical cognition.

As we may recall, Lakoff and Núñez proposed that one of the possible conceptualizations of numbers refers to the idea of points arranged on a line being grounded in bodily experience (Numbers Are Points on the Line). The results of multiple studies pioneered by Dehaene, Bossini, and Giraux (1993) supported this claim by revealing the behavioral tendency of the majority of participants to organize numerical magnitudes spatially. By using the parity judgment paradigm, where the participant is asked to decide whether the presented digit is even or odd, Dehaene and colleagues found that responses of left-to-right readers for small numbers are faster with their left hand, while their RTs in relatively large number trials are shorter with the right hand. This phenomenon is called "Spatial Numerical Association of Response Codes" (SNARC) and has been illustrated on numerous occasions in various study designs and groups of participants (see Cipora, Hohol, Nuerk, Willmes, Brożek, Kucharzyk, & Nęcka, 2016; Cipora, Soltanlou, Reips, & Nuerk, 2019; G. Wood, Willmes, & Nuerk, 2008). As Deahene (2011) summarized this phenomenon in his widely read book entitled *The Number Sense*:

> The finding of an automatic association between numbers and space leads to a simple yet remarkably powerful metaphor for the mental representation of numerical quantities: that of a number line. It is as if numbers were mentally aligned on a segment, with each location corresponding to a certain quantity. Close numbers are represented at adjoining locations. No wonder, then, that we tend to confound them, as reflected by the numerical distance effect. Furthermore, the line can be metaphorically thought of as being oriented in space: Zero is at the extreme left, with larger numbers extending toward the right. This is why the reflex encoding of Arabic numerals as quantities is also accompanied by an automatic orientation of numbers in space, small ones to the left and large ones to the right. (p. 70)

At this point it is worth saying that the title of the above-quoted work is far from accidental. Although the term *number sense* was coined much earlier by Tobias Dantzig (1954), the title of Dehaene's book reflects the contemporary emphasis on studying the *hardwired* cognitive capacities of elementary number processing, both ontogenetically early and evolutionary ancient, that precede the learning of school mathematics and which are shared by all people around the world (Pica, Lemer, Izard, & Dehaene, 2004) and some nonhuman animals (Biro & Matzusawa, 2001). These capacities are considered domain-specific, since they are independent of general cognitive factors such as fluid intelligence, executive functions, or linguistic skills, and involve subtilizing (i.e., ability to immediately and effortlessly assess the number of small collections with a high degree of precision) and the estimation and comparison of larger-magnitude sets of elements (see Berch, 2005; Hohol, Cipora, Willmes, & Nuerk, 2017). Although there is still some debate regarding this matter, Dehaene and many other researchers believe that arithmetic and symbolic numerical systems, which are unquestionably cultural inventions, are built up or scaffold themselves on the number sense (Butterworth, 2005; Dehaene, 2001; Feigenson, Dehaene, & Spelke, 2004; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). There is also a suggestion that the number sense, or the hardwired foundations of mathematics in general, may be interpreted as a renewed version of Kant's views (Dehaene & Brannon, 2010).

Let us return to the SNARC effect. Although Dehaene originally explained the direction of the mental number line by the direction of reading,[18] Fischer (2008) has proposed that spatial-numerical associations are shaped during individual development prior to the acquisition of reading and that the direction of finger counting affects these associations (see also Patro & Haman, 2012). The crucial observation supporting this hypothesis is that participants who start counting with their left hand manifest more robust and consistent spatial-numerical associations than right-starters (see Cipora, Patro, & Nuerk, 2015; Fischer & Brugger, 2011;

Hohol, Wołoszyn, Nuerk, & Cipora, 2018 for the discussion of this claim). Since the beginning of the twenty-first century, finger counting ceased to be perceived merely as a transitory step, or immature computational strategy appearing in the ontogeny of mathematical skills, and instead became the most extensively studied manifestation of the embodiment of numerical cognition (see e.g., Fischer, 2012; Jordan, Kaplan, Ramineni, & Locuniak, 2008; Noël, 2005; Penner-Wilger & M. L. Anderson, 2013; Soylu, Lester, & Newman, 2018; Wołoszyn & Hohol, 2017). Furthermore, the embodied perspective of numerical cognition, with finger counting to the fore, has been considered with reference to anthropological data (Overmann, 2014) and cultural differences (Lindemann, Alipour, & Fischer, 2011), as well as situational influences and cognitive flexibility (Hohol et al., 2018; Wasner, Moeller, Fischer, & Nuerk, 2014).

The set of evidence supporting the view that finger counting and number processing are deeply connected includes not only the results of behavioral experiments but also neuroimaging data. For instance, in a study with positron emission tomography (PET), Zago and colleagues (2001) found that both performing simple mental calculations and ordinary finger-related actions, such as learning of movement patterns and manipulating objects, involve the activations of the same brain structures, including the parieto-premotor circuit. A subsequent functional magnetic resonance imaging (fMRI) study by Tschentscher, Hauk, Fischer, and Pulvermüller (2012) revealed that finger-counting routines regarding the starting hand affect the pattern of motor cortex activation. Finally, Andres, Seron, and Olivier (2007) proposed that the relationship between finger counting and elementary numerical cognition has a causal character since the numbers are cortically processed through the embodied simulation of finger movements (we will discuss the notion of "simulation" and other critical issues of embodied cognition in Chapter 3).

The purpose of the above elaboration was not to provide a sketch of the landscape or to review the theoretical stances of the cognitive science of mathematics (this term was popularized by Lakoff and Núñez). This task would be impossible here for practical reasons. My intention was rather to illustrate that during the development of cognitive studies on mathematics, accompanied by theoretical progress regarding the role of the body in the shaping of thinking, "the center of mass" of research shifted toward the processing of numbers and calculations. This observation can be further substantiated with the following bibliographical and institutional facts.

If one picks up any mathematics-related cognitive science book monographs, handbooks, or collection of articles, it is clear that they focus solely, or in the vast majority, on number processing. In some cases, the content is fairly reflected by the title, as in the above-cited work of Dehaene (2011), *The Number Sense*, and in other literature positions (Butterworth, 1999; Cohen Kadosh & Dowker, 2015; C. Everett, 2017; Geary et al., 2015; Henik, 2016). In other instances, the titles suggest a broader range of topics because of the use of terms "mathematics" or "mathematical cognition" (e.g., Adams, Barmby, & Mesoudi, 2017; Berch et al., 2016; Brożek & Hohol, 2017; Campbell, 2005; Gilmore, Göbel, & Inglis, 2018; Lakoff & Núñez, 2000; Saxe, 2014). In practice, these volumes are primarily focused on numbers, and geometry is a marginal topic.[19] For instance, in Lakoff's and Núñez's *Where Mathematics Comes From*, which aspires to be the most comprehensive account, the discussion of geometry is restricted only to the Cartesian plane and analytic geometry, and the cognitive foundations of Euclidean approach are absent. Admittedly, there are few books where both classic studies and theories of Euclidean thinking—with which we are already familiar—and more recent ones—which we will discuss in the following chapters—are reviewed (Dehaene & Brannon, 2011; Geary, Berch, Ochsendorf, & Koepke, 2017; Giaquinto, 2007; Roth, 2011). This does not, however, change the general observation that geometric cognition is in the minority.

An analysis of the institutional basis of the cognitive science of mathematics also shows that numbers and calculations constitute a research topic that overshadows geometry. On the one hand, studies on both numerical and geometric cognition are welcome in almost all cognitive science (or their constituent branches) journals and conference meetings. On the other hand, one of the signs of research specialization is the creation of journals focused on publishing original research, reviews, or theoretical contributions, on specific topics. Indeed, in 1995, the specialist journal entitled *Mathematical Cognition* was established, but only four years later it closed because of the insufficient number of submissions. Perhaps symptomatic of this, none of the articles published in the journal during its short existence concerned geometric cognition. The same is true regarding the *Journal of Numerical Cognition*, because of its clearly defined specialization, and which has been publishing since 2015. A similar sentiment can be expressed about symposia and conferences: in contrast to the processing of numbers, there is no cyclical conference regarding geometric cognition.

Fortunately, geometry-related cognitive studies, as we will see in subsequent chapters, reach beyond those discussed so far. It is hard to deny, however, that although cognitive science has been interested in how geometric thinking works almost from the outset, over time the mainstream of cognitive studies began to focus increasingly on number processing. When the term *cognitive science of mathematics* became popular, most of the academic milieu identified it with studying how the mind deals with numbers. Traversing Freudenthal's (1971) summary of the long-term relationship between geometry and mathematics quoted at the end of Section 1.2, we can say that for the majority of cognitive scientists today, mathematical cognition is synonymous with number processing.

## 1.7 Summary

I have described different perspectives on studying geometric thinking in the sections above. I began from the perspective of the history of mathematics, trying to show the evolution of geometry from the art of measuring and practical knowledge about figures or polyhedrons to the paradise of abstraction, where geometric reasonings are necessarily true and lead to general conclusions. I then adopted a philosophical perspective and presented a timeless discussion running from Plato to Helmholtz on the sources of geometric knowledge. Afterward, I discussed the first psychological and experimentally grounded account of geometry by Piaget and Inhelder, which considered its subject matter from the perspective of cognitive development. Subsequently, we moved on to an educational perspective on the development of geometric skills, and the classic model by van Hieles in particular, which inspired numerous curricula and continues to do so today. Finally, in the last section, I covered the cognitive revolution of the 1950s and investigated how studies on mathematical cognition have changed over time. The last piece familiarized us with a research perspective that we will explore—however with some recursions to those presented in the previous sections—until the end of the book. In the following chapter, we will look at the hardwired foundations of geometric cognition.

## References


Abrahamsen, A., & Bechtel, W. 2015. Diagrams as tools for scientific reasoning. *Review of Philosophy and Psychology*, *6*(1), 117–131.

Adams, J., Barmby, P., & Mesoudi, A. (Eds.). 2017. *The nature and development of mathematics: Cross-disciplinary perspectives on cognition, learning and culture*. New York: Routledge.

Adorni, R., & Proverbio, A. M. 2012. The neural manifestation of the word concreteness effect: An electrical neuroimaging study. *Neuropsychologia*, *50*(5), 880–891.

Alderson-Day, B., & Fernyhough, C. 2015. Inner speech: Development, cognitive functions, phenomenology, and neurobiology. *Psychological Bulletin*, *141*(5), 931–965.

Alyahya, R. S. W., Halai, A. D., Conroy, P., & Ralph, M. A. L. 2018. The behavioural patterns and neural correlates of concrete and abstract verb processing in aphasia: A novel verb semantic battery. *NeuroImage: Clinical*, *17*, 811–825.

Amalric, M., & Dehaene, S. 2016. Origins of the brain networks for advanced mathematics in expert mathematicians. *Proceedings of the National Academy of Sciences*, *113*(18), 4909–4917.

Amalric, M., & Dehaene, S. 2019. A distinct cortical network for mathematical knowledge in the human brain. *NeuroImage*, *189*, 19–31.

Amalric, M., Wang, L., Pica, P., Figueira, S., Sigman, M., & Dehaene, S. 2017. The language of geometry: Fast comprehension of geometrical primitives and rules in human adults and preschoolers. *PLOS Computational Biology*, *13*(1), e1005273.

Anderson, J. R. 1983. *The architecture of cognition*. New York: Psychology Press.

Anderson, J. R., Boyle, C. F., & Yost, G. 1985. The geometry tutor. In *Proceedings of the International Joint Conference on Artificial Intelligence* (Vol. *85*). Los Angeles: IJCAI.

Anderson, M., & McCartney, R. 2003. Diagram processing: Computing with diagrams. *Artificial Intelligence*, *145*(1–2), 181–226.

Anderson, M. L. 2003. Embodied cognition: A field guide. *Artificial Intelligence*, *149*(1), 91–130.

Anderson, M. L. 2010. Neural reuse: A fundamental organizational principle of the brain. *Behavioral and Brain Sciences*, *33*(4), 245–266.

Anderson, M. L. 2014. *After phrenology: Neural reuse and the interactive brain*. Cambridge, MA: The MIT Press.

Andres, M., Seron, X., & Olivier, E. 2007. Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience*, *19*(4), 563–576.

Angelelli, I. 2004. Adventures of abstraction. *Poznań Studies in the Philosophy of the Sciences and Humanities*, *82*, 11–35.

Aristotle. 2009a. *Metaphysics*. (W. D. Ross, Trans.). Cambridge, MA: The MIT: The Internet Classics Archive. Retrieved from http://classics.mit.edu/Aristotle/metaphysics.html.

Aristotle. 2009b. *Posterior analytics*. (G. R. G. Mure, Trans.). Cambridge, MA: The MIT: The Internet Classics Archive. Retrieved from http://classics.mit.edu/Aristotle/posterior.html.

Aufrere, S. H., & Marganne, M.-H. 2019. Encounters between Greek and Egyptian science. In K. Vandorpe (Ed.), *A companion to Greco-Roman and late Antique Egypt* (pp. 501–518). New York: Wiley.

Avargues-Weber, A., Dyer, A. G., Combe, M., & Giurfa, M. 2012. Simultaneous mastering of two abstract concepts by the miniature brain of bees. *Proceedings of the National Academy of Sciences*, *109*(19), 7481–7486.

Avargues-Weber, A., & Giurfa, M. 2013. Conceptual learning by miniature brains. *Proceedings of the Royal Society B: Biological Sciences*, *280*(1772), 20131907.

Avigad, J., Dean, E., & Mumma, J. 2009. A formal system for Euclid's elements. *Review of Symbolic Logic*, *2*(4), 700–768.

Aziz-Zadeh, L., Wilson, S. M., Rizzolatti, G., & Iacoboni, M. 2006. Congruent embodied representations for visually presented actions and linguistic phrases describing actions. *Current Biology*, *16*(18), 1818–1823.

Azzouni, J. 2004. The derivation-indicator view of mathematical practice. *Philosophia Mathematica*, *12*(2), 81–106.

Bardi, J. S. 2008. *The fifth postulate*. Hoboken, NJ: John Wiley & Sons.

Bardone, E., & Magnani, L. 2007. Sharing representations through cognitive niche construction. *Data Science Journal*, *6*, S87–S91.

Barkow, J., Cosmides, L., & Tooby, J. (Eds.). 1992. *The adapted mind: Biological approaches to mind and culture*. Oxford, UK: Oxford University Press.

Barsalou, L. W. 1999. Perceptual symbol systems. *Behavioral and Brain Sciences*, *22*(4), 577–660.

Barsalou, L. W. 2008. Grounded cognition. *Annual Review of Psychology*, *59*, 617–645.

Barsalou, L. W., Dutriaux, L., & Scheepers, C. 2018. Moving beyond the distinction between concrete and abstract concepts. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *373*(1752), 20170144–20170154.

Barsalou, L. W., Simmons, W. K., Barbey, A. K., & Wilson, C. D. 2003. Grounding conceptual knowledge in modality-specific systems. *Trends in Cognitive Sciences*, 7(2), 84–91.

Barsalou, L. W., & Wiemer-Hastings, K. 2005. Situating abstract concepts. In D. Pecher, & R. Zwaan (Eds.), *Grounding cognition the role of perception and action in memory, language and thinking* (pp. 129–163). Cambridge, UK: Cambridge University Press.

Bateson, P., & Laland, K. N. 2013. Tinbergen's four questions: An appreciation and an update. *Trends in Ecology & Evolution*, *28*(12), 712–718.

Battista, M. T. 2007. The development of geometric and spatial thinking. In F. K. Lester, (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). New York: Information Age Publishing.

Bechtel, W. 2008. *Mental mechanisms: Philosophical perspectives on cognitive neuroscience*. New York: Routledge.

Bechtel, W. 2016. Investigating neural representations: The tale of place cells. *Synthese*, *193*(5), 1287–1321.

Bechtel, W., Abrahamsen, A., & Graham, G. 1998. The life of cognitive science. In W. Bechtel, & G. Graham (Eds.), *A companion to cognitive science* (pp. 2–104). Malden, MA: Blackwell Publishers.

Bechtold, L., Bellebaum, C., Egan, S., Tettamanti, M., & Ghio, M. 2019. The role of experience for abstract concepts: Expertise modulates the electrophysiological correlates of mathematical word processing. *Brain and Language*, *188*, 1–10.

Benton, M. J. 2015. *Vertebrate palaeontology* (4th ed.). Oxford, UK: Wiley Blackwell.

Berch, D. B. 2005. Making sense of number sense: Implications for children with mathematical disabilities. *Journal of Learning Disabilities*, *38*(4), 333–339.

Berch, D. B., Geary, D. C., & Koepke, K. M. 2016. *Development of mathematical cognition: Neural substrates and genetic influences*. Amsterdam: Academic Press.

Berendt, B., Barkowsky, T., Freksa, C., & Kelter, S. 1998. Spatial representation with aspect maps. In C. Freksa, C. Habel, & K. F. Wender (Eds.), *Spatial cognition: An interdisciplinary approach to representing and processing spatial knowledge* (pp. 313–336). Berlin: Springer.

Berthold, P. (Ed.). 1991. *Orientation in birds*. Basel, Switzerland: Birkhauser.

Biagioli, F. 2013. What does it mean that "Space can be transcendental without the axioms being so?". *Journal for General Philosophy of Science*, *45*(1), 1–21.

Biagioli, F. 2016. *Space, number, and geometry from Helmholtz to Cassirer* (Vol. *46*). Cham, Switzerland: Springer.

Binder, J. R. 2016. In defense of abstract conceptual representations. *Psychonomic Bulletin & Review*, *23*(4), 1–13.

Binder, J. R., & Desai, R. H. 2011. The neurobiology of semantic memory. *Trends in Cognitive Sciences*, *15*(11), 527–536.

Binder, J. R., Desai, R. H., Graves, W. W., & Conant, L. L. 2009. Where is the semantic system? A critical review and meta-analysis of 120 functional neuroimaging studies. *Cerebral Cortex*, *19*(12), 2767–2796.

Biro, D., & Matsuzawa, T. 2001. Chimpanzee numerical competence: Cardinal and ordinal skills. In T. Matzusawa (Ed.), *Primate origins of human cognition and behavior* (pp. 199–225). Tokyo: Springer.

Blaut, J. M., McCleary, G. F., & Blaut, A. S. 1970. Environmental mapping in young children. *Environment and Behavior*, *2*(3), 335–349.

Block, N. J., & Fodor, J. A. 1972. What psychological states are not. *The Philosophical Review*, *81*(2), 159–181.

Bluestein, N., & Acredolo, L. 1979. Developmental changes in map-reading skills. *Child Development*, *50*(3), 691–697.

Borghi, A. M., Barca, L., Binkofski, F., & Tummolini, L. 2018. Varieties of abstract concepts: Development, use and representation in the brain. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, *373*(1752), 20170121–20170127.

Borghi, A. M., & Binkofski, F. 2014. *Words as social tools: An embodied view on abstract concepts*. New York: Springer.

Borghi, A. M., Scorolli, C., Caligiore, D., Baldassarre, G., & Tummolini, L. 2013. The embodied mind extended: Using words as social tools. *Frontiers in Psychology*, *4*(214), 1–10.

Boroditsky, L. 2001. Does language shape thought? Mandarin and English speakers' conceptions of time. *Cognitive Psychology*, *43*(1), 1–22.

Bowen, A. C. 1983. Menaechmus versus the Platonists: Two theories of science in the early academy. *Ancient Philosophy*, *3*(1), 12–29.

Brown, A. A., Spetch, M. L., & Hurd, P. L. 2007. Growing in circles: Rearing environment alters spatial navigation in fish. *Psychological Science*, *18*(7), 569–573.
Brożek, B. 2013. *Rule-following: From imitation to the normative mind*. Kraków, Poland: Copernicus Center Press.
Brożek, B., & Hohol, M. 2017. *Umysł matematyczny* (The mathematical mind) (3rd ed.). Kraków, Poland: Copernicus Center Press.
Brożek, B., & Olszewski, A. 2011. The mathematics of the transcendental ego. *Copernicus Center Reports*, *2*, 75–124.
Bruner, J. S., Goodnow, J. J., & Austin, G. A. 1956. *A study of thinking*. New York: Wiley.
Buehlmann, C., Graham, P., Hansson, B. S., & Knaden, M. 2015. Desert ants use olfactory scenes for navigation. *Animal Behaviour*, *106*, 99–105.
Buller, D. J. 2006. *Adapting minds: Evolutionary psychology and the persistent quest for human nature*. Cambridge, MA: The MIT Press.
Burger, W. F., & Shaughnessy, J. M. 1986. Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, *17*(1), 31.
Buss, D. M. 2009. Evolutionary psychology: A new paradigm for psychological science. *Psychological Inquiry*, *6*(1), 1–30.
Butterworth, B. 1999. *What counts: How every brain is hardwired for math*. New York: Free Press.
Butterworth, B. 2005. The development of arithmetical abilities. *Journal of Child Psychology and Psychiatry*, *46*(1), 3–18.
Cajori, F. 1928. *A history of mathematical notations* (Vol. *1*). London: Open Court.
Calero, C. I., Shalom, D. E., Spelke, E. S., & Sigman, M. 2019. Language, gesture, and judgment: Children's paths to abstract geometry. *Journal of Experimental Child Psychology*, *177*, 70–85.
Campbell, J. I. D. 2005. *Handbook of mathematical cognition*. New York: Psychology Press.
Carey, S. C. 2009. *The origin of concepts*. Oxford, UK: Oxford University Press.
Carey, S. C., & Spelke, E. S. 1996. Science and core knowledge. *Philosophy of Science*, *63*(4), 515–533.
Carruthers, P. 2006. *The architecture of the mind*. Oxford, UK: Oxford University Press.
Cartwright, B. A., & Collett, T. S. 1983. Landmark learning in bees. *Journal of Comparative Physiology A: Sensory, Neural, and Behavioral Physiology*, *151*(4), 521–543.
Casasanto, D., & Boroditsky, L. 2008. Time in the mind: Using space to think about time. *Cognition*, *106*(2), 579–593.
Castellanos, J. A., & Tardos, J. D. 1999. *Mobile robot localization and map building*. New York: Springer.
Cavalli-Sforza, L. L., & Feldman, M. W. 1981. *Cultural transmission and evolution: A quantitative approach*. Princeton, NJ: Princeton University Press.
Chatterjee, A. 2010. Disembodying cognition. *Language and Cognition*, *2*(1), 79–116.
Chemero, A. 2011. *Radical embodied cognitive science*. Cambridge, MA: The MIT Press.
Cheng, K. 1986. A purely geometric module in the rat's spatial representation. *Cognition*, *23*(2), 149–178.
Cheng, K. 2008. Whither geometry? Troubles of the geometric module. *Trends in Cognitive Sciences*, *12*(9), 355–361.
Cheng, K., Huttenlocher, J., & Newcombe, N. S. 2013. 25 years of research on the use of geometry in spatial reorientation: A current theoretical perspective. *Psychonomic Bulletin & Review*, *20*(6), 1033–1054.
Cheng, K., & Newcombe, N. S. 2005. Is there a geometric module for spatial orientation? Squaring theory and evidence. *Psychonomic Bulletin & Review*, *12*(1), 1–23.

Cheng, K., Shettleworth, S. J., Huttenlocher, J., & Rieser, J. J. 2007. Bayesian integration of spatial information. *Psychological Bulletin*, *133*(4), 625–637.

Chiappe, D., & MacDonald, K. 2005. The evolution of domain-general mechanisms in intelligence and learning. *The Journal of General Psychology*, *132*(1), 5–40.

Chomsky, N. 1965. *Aspects of the theory of syntax*. Cambridge, MA: The MIT Press.

Chomsky, N. 1980. *Rules and representations*. New York: Columbia University Press.

Chomsky, N. 2006. *Language and mind*. Cambridge, UK: Cambridge University Press.

Chomsky, N. 2009. *Cartesian linguistics*. Cambridge, UK: Cambridge University Press.

Cipora, K., Hohol, M., Nuerk, H.-C., Willmes, K., Brożek, B., Kucharzyk, B., & Nęcka, E. 2016. Professional mathematicians differ from controls in their spatial-numerical associations. *Psychological Research*, *80*(4), 710–726.

Cipora, K., Patro, K., & Nuerk, H.-C. 2015. Are spatial-numerical associations a cornerstone for arithmetic learning? The lack of genuine correlations suggests no. *Mind, Brain, and Education*, *9*(4), 190–206.

Cipora, K., Soltanlou, M., Reips, U.-D., & Nuerk, H.-C. 2019. The SNARC and MARC effects measured online: Large-scale assessment methods in flexible cognitive effects. *Behavior Research Methods*, 1–17. Online first. Retrieved from https://doi.org/10.3758/s13428-019-01213-5

Clark, A. 1998. *Being there: Putting brain, body, and world together again*. Cambridge, MA: The MIT Press.

Clark, A. 2003. *Natural-born cyborgs: Minds, technologies, and the future of human intelligence*. Oxford, UK: Oxford University Press.

Clark, A. 2006. Language, embodiment, and the cognitive niche. *Trends in Cognitive Sciences*, *10*(8), 370–374.

Clark, A. 2008. *Supersizing the mind: Embodiment, action, and cognitive extension*. Oxford, UK: Oxford University Press.

Clements, D. H., & Battista, M. T. 1992. Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Information Age Publishing.

Clements, D. H., & Sarama, J. 2011. Early childhood teacher education: The case of geometry. *Journal of Mathematics Teacher Education*, *14*(2), 133–148.

Cohen Kadosh, R., & Dowker, A. (Eds.). 2015. *The Oxford handbook of numerical cognition*. Oxford, UK: Oxford University Press.

Cohen-Kadosh, R., Lammertyn, J., & Izard, V. 2008. Are numbers special? An overview of chronometric, neuroimaging, developmental and comparative studies of magnitude representation. *Progress in Neurobiology*, *84*(2), 132–147.

Connell, L., & Lynott, D. 2012. Strength of perceptual experience predicts word processing performance better than concreteness or imageability. *Cognition*, *125*(3), 452–465.

Cooper, L. A., & Shepard, R. N. 1973. Chronometric studies of the rotation of mental images. In W. G. Chase (Ed.), *Visual information processing: Proceedings* (pp. 75–176). New York: Academic Press.

Corry, L. 2004. *David Hilbert and the axiomatization of physics (1898–1918): From Grundlagen der Geometrie to Grundlagen der Physik*. Dordrecht, Netherlands: Springer.

Cosmides, L., & Tooby, J. 1997. *Evolutionary psychology: A primer*. Retrieved from http://infantcognitiongroup.com/Portals/1/Cosmides%26Tooby(1997).pdf.

Costello, F. J., & Keane, M. T. 2000. Efficient creativity: Constraint-guided conceptual combination. *Cognitive Science*, *24*(2), 299–349.

Cousins, K. A. Q., York, C., Bauer, L., & Grossman, M. 2016. Cognitive and anatomic double dissociation in the representation of concrete and abstract words in semantic variant and behavioral variant frontotemporal degeneration. *Neuropsychologia*, *84*, 244–251.

Craver, C. F. 2009. *Explaining the brain: Mechanisms and the mosaic unity of neuroscience*. Oxford, UK: Oxford University Press.

Craver, C. F., & Darden, L. 2013. *In search of mechanisms*. Chicago: University of Chicago Press.

Craver, C. F., & Kaplan, D. M. 2018. Are more details better? On the norms of completeness for mechanistic explanations. *The British Journal for the Philosophy of Science*. Online first. Retrieved from http://doi.org/10.1093/bjps/axy015.

Cruse, H., & Wehner, R. 2011. No need for a cognitive map: Decentralized memory for insect navigation. *PLOS Computational Biology*, 7(3), e1002009.

Csordas, T. J. 1990. Embodiment as a paradigm for anthropology. *Ethos*, *18*(1), 5–47.

Curio, E. 1973. Towards a methodology of teleonomy. *Experientia*, *29*(9), 1045–1058.

Damasio, A. R. 1989. The brain binds entities and events by multiregional activation from convergence zones. *Neural Computation*, *1*(1), 123–132.

Dantzig, T. 1954. *Number: The language of science*. New York: MacMillan.

Darke, I. 1982. A review of research related to the topological primacy thesis. *Educational Studies in Mathematics*, *13*(2), 119–142.

Davis, J. I., & Markman, A. B. 2012. Embodied cognition as a practical paradigm: Introduction to the topic, the future of embodied cognition. *Topics in Cognitive Science*, *4*(4), 685–691.

Dehaene, S. 1989. The psychophysics of numerical comparison: A reexamination of apparently incompatible data. *Perception & Psychophysics*, *45*(6), 557–566.

Dehaene, S. 1996. The organization of brain activations in number comparison: Event-related potentials and the additive-factors method. *Journal of Cognitive Neuroscience*, *8*(1), 47–68.

Dehaene, S. 2001. Précis of the number sense. *Mind and Language*, *16*(1), 16–36.

Dehaene, S. 2011. *The number sense (Revised)*. Oxford, UK: Oxford University Press.

Dehaene, S., Bossini, S., & Giraux, P. 1993. The mental representation of parity and number magnitude. *Journal of Experimental Psychology General*, *122*(3), 371–396.

Dehaene, S., & Brannon, E. M. 2010. Space, time, and number: A Kantian research program. *Trends in Cognitive Sciences*, *14*(12), 517–519.

Dehaene, S., & Brannon, E. M. (Eds.). 2011. *Space, time and number in the brain*. Amsterdam: Academic Press.

Dehaene, S., & Cohen, L. 1995. Towards an anatomical and functional model of number processing. *Mathematical Cognition*, *1*, 83–120.

Dehaene, S., & Cohen, L. 1997. Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic. *Cortex*, *33*(2), 219–250.

Dehaene, S., Izard, V., Pica, P., & Spelke, E. S. 2006. Core knowledge of geometry in an Amazonian indigene group. *Science*, *311*(5579), 381–384.

Dehaene, S., Le Clec'H, G., Poline, J.-B., Le Bihan, D., & Cohen, L. 2002. The visual word form area: A prelexical representation of visual words in the fusiform gyrus. *NeuroReport*, *13*(3), 321–325.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. 2003. Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*(3-6), 487–506.

Dehaene, S., Spelke, E. S., Pinel, P., Stanescu, R., & Tsivkin, S. 1999. Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science*, *284*(5416), 970–974.

Deipolyi, A., Santos, L., & Hauser, M. 2001. The role of landmarks in cotton-top tamarin spatial foraging: Evidence for geometric and non-geometric features. *Animal Cognition*, *4*(2), 99–108.

Della Rosa, P. A., Catricalà, E., Vigliocco, G., & Cappa, S. F. 2010. Beyond the abstract—Concrete dichotomy: Mode of acquisition, concreteness, imageability, familiarity, age of acquisition, context availability, and abstractness norms for a set of 417 Italian words. *Behavior Research Methods*, *42*(4), 1042–1048.

Dennett, D. C. 1981. *Brainstorms*. Cambridge, MA: The MIT Press.

Derdikman, D., & Moser, E. I. 2010. A manifold of spatial maps in the brain. *Trends in Cognitive Sciences*, *14*(12), 561–569.

De Risi, V. (Ed.). 2015. *Mathematizing space: The objects of geometry from antiquity to the early modern age*. Heidelberg, Germany: Birkhauser.

Descartes, R. 1976. *Conversation with Burman*. Oxford, UK: Clarendon Press.

Descartes, R. 1998. *Regulae ad directionem ingenii/Rules for the direction of the natural intelligence: A bilingual edition* (G. Heffernan, Trans.). Amsterdam: Rodopi.

Descartes, R. 2012. *The geometry of Rene Descartes*. (D. E. Smith & L. A. Latham, Trans.). New York: Dover Publications.

Detlefsen, M. 2005. Formalism. In S. Shapiro (Ed.), *The Oxford handbook of philosophy of mathematics and logic* (pp. 236–317). Oxford, UK: Oxford University Press.

D'Hoine, P., & Martijn, M. (Eds.). 2017. *All from one: A guide to Proclus*. Oxford, UK: Oxford University Press.

Dilks, D. D., Julian, J. B., Paunov, A. M., & Kanwisher, N. 2013. The occipital place area is causally and selectively involved in scene perception. *The Journal of Neuroscience*, *33*(4), 1331–1336.

Dillon, M. R., Persichetti, A. S., Spelke, E. S., & Dilks, D. D. 2017. Places in the brain: Bridging layout and object geometry in scene-selective cortex. *Cerebral Cortex*, *28*, 2365–2374.

Dillon, M. R., & Spelke, E. S. 2018. From map reading to geometric intuitions. *Developmental Psychology*, *54*(7), 1304–1316.

Dingemanse, M., Blasi, D. E., Lupyan, G., Christiansen, M. H., & Monaghan, P. 2015. Arbitrariness, iconicity, and systematicity in language. *Trends in Cognitive Sciences*, *19*(10), 603–615.

Doeller, C. F., Barry, C., & Burgess, N. 2010. Evidence for grid cells in a human memory network. *Nature*, *463*(7281), 657–661.

Dove, G. 2009. Beyond perceptual symbols: A call for representational pluralism. *Cognition*, *110*(3), 412–431.

Dove, G. 2011. On the need for embodied and dis-embodied cognition. *Frontiers in Psychology*, *1*(242), 1–13.

Dove, G. 2014. Thinking in words: Language as an embodied medium of thought. *Topics in Cognitive Science*, *6*(3), 371–389.

Dove, G. 2016. Three symbol ungrounding problems: Abstract concepts and the future of embodied cognition. *Psychonomic Bulletin & Review*, *23*(4), 1109–1121.

Dove, G. 2018. Language as a disruptive technology: Abstract concepts, embodiment and the flexible mind. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *373*(1752), 20170135–20170143.

Druks, J., Masterson, J., Kopelman, M., Clare, L., Rose, A., & Rai, G. 2006. Is action naming better preserved (than object naming) in Alzheimer's disease and why should we ask? *Brain and Language*, *98*(3), 332–340.

Duval, A. 2019. The representation selection problem: Why we should favor the geometric-module framework of spatial reorientation over the view-matching framework. *Cognition*, *192*, 1–20.

Dyer, F. C. 1991. Bees acquire route-based memories but not cognitive maps in a familiar landscape. *Animal Behaviour*, *41*(2), 239–246.

Dzbyński, A. 2014. *The power of the line: Metaphor, number and material culture in European prehistory*. Cambridge, UK: Cambridge Scholars Publishing.

Egan, K. 1983. *Education and psychology: Plato, Piaget and scientific psychology*. New York: Routledge.

Ericsson, K. A., & Simon, H. A. 1984. *Protocol analysis: Verbal reports as data*. Cambridge, MA: The MIT Press.

Everett, C. 2017. *Numbers and the making of us*. Cambridge, MA: Harvard University Press.

Everett, D. L. 2012. *Language: The cultural tool*. London: Profile Books.

Feigenson, L., Dehaene, S., & Spelke, E. S. 2004. Core systems of number. *Trends in Cognitive Sciences*, *8*(7), 307–314.

Fernandino, L., Binder, J. R., Desai, R. H., Pendl, S. L., Humphries, C. J., Gross, W. L., Conant, L. L., & Seidenberg, M. 2016. Concept representation reflects multimodal abstraction: A framework for embodied semantics. *Cerebral Cortex*, *26*(5), 2018–2034.

Fernandino, L., Conant, L. L., Binder, J. R., Blindauer, K., Hiner, B., Spangler, K., & Desai, R. H. 2013. Where is the action? Action sentence processing in Parkinson's disease. *Neuropsychologia*, *51*(8), 1510–1517.

Fingerhut, J., & Prinz, J. J. 2018. Grounding evaluative concepts. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *373*(1752), 20170142–20170147.

Fini, C., & Borghi, A. M. 2019. Sociality to reach objects and to catch meaning. *Frontiers in Psychology*, *10*(839), 1–9.

Fischer, M. H. 2008. Finger counting habits modulate spatial-numerical associations. *Cortex*, *44*(4), 386–392.

Fischer, M. H. 2012. A hierarchical view of grounded, embodied, and situated numerical cognition. *Cognitive Processing*, *13*(S1), 161–164.

Fischer, M. H., & Brugger, P. 2011. When digits help digits: Spatial, numerical associations point to finger counting as prime example of embodied cognition. *Frontiers in Psychology*, *2*(260), 1–7.

Fischer, M. H., & Zwaan, R. A. 2008. Embodied language: A review of the role of the motor system in language comprehension. *The Quarterly Journal of Experimental Psychology*, *61*(6), 825–850.

Fitzpatrick, R. (Ed.). 2008. *Euclid's Elements of geometry from Euclidis Elementa by I. L. Heiberg (1883)*. Retrieved from http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf.

Flavell, J. H. 1963. *The developmental psychology of Jean Piaget*. Princeton, NJ: Van Nostrand.

Fodor, J. A. 1975. *The language of thought*. Cambridge, MA: Harvard University Press.

Fodor, J. A. 1983. *The modularity of mind*. Cambridge, MA: The MIT Press.

Fodor, J. A. 1992. *A theory of content and other essays*. Cambridge, MA: Bradford Books.

Freksa, C., & Barkowsky, T. 1999. On the duality and on the integration of propositional and spatial representations. *Advances in Psychology*, *128*, 195–212.

Freksa, C., Barkowsky, T., Falomir, Z., & van de Ven, J. 2019. Geometric problem solving with strings and pins. *Spatial Cognition & Computation*, *19*(1), 46–68.

Freudenthal, H. 1971. Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3–4), 413–435.

Fuchs, T., & Schlimme, J. E. 2009. Embodiment and psychopathology: A phenomenological perspective. *Current Opinion in Psychiatry*, *22*(6), 570–575.

Fumarola, A., Prpic, V., Fornasier, D., Sartoretto, F., Agostini, T., & Umiltà, C. 2016. The spatial representation of angles. *Perception*, *45*(11), 1320–1330.
Fuys, D., Geddes, D., & Tischler, R. 1988. *The Van Hiele model of thinking in geometry among adolescents*. Reston, VA: National Council of Teachers of Mathematics.
Fyhn, M., Molden, S., Witter, M. P., Moser, E. I., & Moser, M.-B. 2014. Spatial representation in the entorhinal cortex. *Science*, *305*, 1258–1264.
Gallagher, S. 2017. *Enactivist interventions: Rethinking the mind*. Oxford, UK: Oxford University Press.
Gallese, V., & Lakoff, G. 2005. The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*(3–4), 455–479.
Gallistel, C. 1980. *The organization of action: A new synthesis*. Hillsdale, NJ: Lawrence Erlbaum.
Gallistel, C. 1989. Animal cognition: The representation of space, time and number. *Annual Review of Psychology*, *40*(1), 155–189.
Gallistel, C. 1990. *The organization of learning*. Cambridge, MA: The MIT Press.
Ganis, G., Thompson, W. L., & Kosslyn, S. M. 2004. Brain areas underlying visual mental imagery and visual perception: An fMRI study. *Cognitive Brain Research*, *20*(2), 226–241.
Gärdenfors, P. 2004. *Conceptual spaces: The geometry of thought*. Cambridge, MA: The MIT Press.
Gärdenfors, P. 2014. *The geometry of meaning: Semantics based on conceptual spaces*. Cambridge, MA: The MIT Press.
Gardner, H. 1985. *The mind's new science: A history of cognitive revolution*. New York: Basic Books.
Geary, D. C., Berch, D. B., & Koepke, K. M. 2015. *Evolutionary origins and early development of number processing*. Amsterdam: Academic Press.
Geary, D. C., Berch, D. B., Ochsendorf, R., & Koepke, K. M. (Eds.). 2017. *Acquisition of complex arithmetic skills and higher-order mathematics concepts*. Amsterdam: Academic Press.
Gee, A. P., Chekhlov, D., Calway, A., & Mayol-Cuevas, W. 2008. Discovering higher level structure in visual SLAM. *IEEE Transactions on Robotics*, *24*(5), 980–990.
Gelernter, H. 1963. Realization of a geometry-theorem proving machine. In E. Feigenbaum, & J. Feldman (Eds.), *Computers and thought* (pp. 134–153). New York: McGraw Hill.
Ghio, M., Vaghi, M. M. S., Perani, D., & Tettamanti, M. 2016. Decoding the neural representation of fine-grained conceptual categories. *NeuroImage*, *132*, 93–103.
Giaquinto, M. 1992. Visualizing as a means of geometrical discovery. *Mind and Language*, *7*(4), 382–401.
Giaquinto, M. 2007. *Visual thinking in mathematics*. Oxford, UK: Oxford University Press.
Gibbs, R. 1996. Why many concepts are metaphorical. *Cognition*, *61*, 309–319.
Gibbs, R. 2005. *Embodiment and cognitive science*. Cambridge, UK: Cambridge University Press.
Gibson, E. J. 1969. *Principles of perceptual learning and development*. New York: Appleton Century Crofts.
Gibson, J. J. 2015. *The ecological approach to visual perception*. New York: Psychology Press.
Giesbrecht, B., Camblin, C. C., & Swaab, T. Y. 2004. Separable effects of semantic priming and imageability on word processing in human cortex. *Cerebral Cortex*, *14*(5), 521–529.
Gilmore, C. K., Göbel, S. M., & Inglis, M. 2018. *An introduction to mathematical cognition*. New York: Routledge.

Gładziejewski, P. 2015. Explaining cognitive phenomena with internal representations: A mechanistic perspective. *Studies in Logic, Grammar and Rhetoric*, *40*(1), 1–28.

Gładziejewski, P., & Miłkowski, M. 2017. Structural representations: Causally relevant and different from detectors. *Biology and Philosophy*, *32*(3), 337–355.

Gleick, J. 2011. *Genius: The life and science of Richard Feynman*. New York: Open Road.

Glenberg, A. M. 2010. Embodiment as a unifying perspective for psychology. *Wiley Interdisciplinary Reviews: Cognitive Science*, *42*(4), 586–596.

Glenberg, A. M., & Gallese, V. 2012. Action-based language: A theory of language acquisition, comprehension, and production. *Cortex*, *48*(7), 1–18.

Glenberg, A. M., & Kaschak, M. P. 2002. Grounding language in action. *Psychonomic Bulletin & Review*, *9*(3), 558–565.

Glenberg, A. M., & Robertson, D. A. 1999. Indexical understanding of instructions. *Discourse Processes*, *28*(1), 1–26.

Glenberg, A. M., Sato, M., Cattaneo, L., Riggio, L., Palumbo, D., & Buccino, G. 2008. Processing abstract language modulates motor system activity. *The Quarterly Journal of Experimental Psychology*, *61*(6), 905–919.

Goldberg, R. F., Perfetti, C. A., & Schneider, W. 2006. Distinct and common cortical activations for multimodal semantic categories. *Cognitive, Affective, & Behavioral Neuroscience*, *6*(3), 214–222.

Goldin, G. A. 2001. Counting on the metaphorical. *Nature*, *413*, 18–19.

Goldstone, R. L., & Janssen, M. A. 2005. Computational models of collective behavior. *Trends in Cognitive Sciences*, *9*(9), 424–430.

Goodale, M. A., Króliczak, G., & Westwood, D. A. 2005. Dual routes to action: Contributions of the dorsal and ventral streams to adaptive behavior. *Progress in brain research*, *149*, 269–283.

Goodman, M. K. J. 2016. *An introduction to the early development of mathematics*. Hoboken, NJ: John Wiley & Sons.

Gould, J. L. 1986. The locale map of honey bees: Do insects have cognitive maps? *Science*, *232*(4752), 861–863.

Gouteux, S., Thinus-Blanc, C., & Vauclair, J. 2001. Rhesus monkeys use geometric and nongeometric information during a reorientation task. *Journal of Experimental Psychology. General*, *130*(3), 505–519.

Gouteux, S., Vauclair, J., & Thinus-Blanc, C. 1999. Reaction to spatial novelty and exploratory strategies in baboons. *Animal Learning & Behavior*, *27*(3), 323–332.

Gray, E. R., Spetch, M. L., Kelly, D. M., & Nguen, R. 2004. Searching in the center: Pigeons (Columba livid) encode relative distance from walls of an enclosure. *Journal of Comparative Psychology*, *118*(1), 113–117.

Gray, J. 2012. *Henri Poincare: A scientific biography*. Princeton, NJ: Princeton University Press.

Greenberg, M. J. 1993. *Euclidean and non-Euclidean geometries: Development and history*. New York: W.H. Freeman and Company.

Greeno, J. G. 1978. A study of problem-solving. *Advances in Instructional Psychology*, *1*, 13–75.

Griffiths, P. E. 2009. In what sense does "Nothing make sense except in the light of evolution?" *Acta Biotheoretica*, *57*(1–2), 11–32.

Grill-Spector, K., Golarai, G., & Gabrieli, J. 2008. Developmental neuroimaging of the human ventral visual cortex. *Trends in Cognitive Sciences*, *12*(4), 152–162.

Grill-Spector, K., Kourtzi, Z., & Kanwisher, N. 2001. The lateral occipital complex and its role in object recognition. *Vision Research*, *41*(10–11), 1409–1422.

Gureckis, T. M., & Goldstone, R. L. 2006. Thinking in groups. *Pragmatics & Cognition*, *14*(2), 293–311.

Hadamard, J. 1945. *An essay on the psychology of invention in the mathematical field*. New York: Dover Publications.

Hafting, T., Fyhn, M., Molden, S., Moser, M.-B., & Moser, E. I. 2005. Microstructure of a spatial map in the entorhinal cortex. *Nature*, *436*(7052), 801–806.

Hahn, H. 1980. The crisis in intuition. In B. McGuinness (Ed.), *Empiricism, logic, and mathematics* (pp. 72–102). Dordrecht, Netherlands: Riedel Publishing.

Hardy, G. H. 2005. *A mathematician's apology*. Edmonton, Canada: University of Alberta Mathematical Sciences Society.

Harnad, S. 1990. The symbol grounding problem. *Physica D: Nonlinear Phenomena*, *42*(1–3), 335–346.

Hartshorne, R. 2007. *Geometry: Euclid and beyond*. New York: Springer.

Hatfield, G. C. 1990. *The natural and the normative: Theories of spatial perception from Kant to Helmholtz*. Cambridge, MA: The MIT Press.

Heath, T. 1970. *Mathematics in Aristotle*. Oxford, UK: Clarendon Press.

Heller, M. 2012. Is mathematics poetry? *Copernicus Center Reports*, *3*, 11–17.

Heller, M. 2019. *God and geometry: When space was God*. Kraków, Poland: Copernicus Center Press.

Henik, A. (Ed.). 2016. *Continuous issues in numerical cognition*. London: Academic Press.

Hermer, L., & Spelke, E. S. 1994. A geometric process for spatial orientation in young children. *Nature*, *370*(3), 57–59.

Hermer, L., & Spelke, E. S. 1996. Modularity and development: The case of spatial reorientation. *Cognition*, *61*(3), 195–232.

Hermer-Vazquez, L., Moffet, A., & Munkholm, P. 2001. Language, space, and the development of cognitive flexibility in humans: The case of two spatial memory tasks. *Cognition*, *79*(3), 263–299.

Hermer-Vazquez, L., Spelke, E. S., & Katsnelson, A. S. 1999. Sources of flexibility in human cognition: Dual-task studies of space and language. *Cognitive Psychology*, *39*(1), 3–36.

Hermes, D., Rangarajan, V., Foster, B. L., King, J.-R., Kasikci, I., Miller, K. J., & Parvizi, J. 2017. Electrophysiological responses in the ventral temporal cortex during reading of numerals and calculation. *Cerebral Cortex*, *27*(1), 567–575.

Herodotus. 2009. *The history*. Internet: Project Gutenberg. Retrieved from https://www.gutenberg.org/ebooks/2707.

Hershkowitz, R. 2009. Psychological aspects of learning geometry. In P. Nesher, & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70–95). Cambridge, UK: Cambridge University Press.

Hilbert, D. 1950. *Foundations of geometry*. (E. J. Townsend, Trans.). La Salle, IL: The Open Court Publishing Company.

Hohol, M., Baran, B., Krzyżowski, M., & Francikowski, J. 2017a. Does spatial navigation have a blind-spot? Visiocentrism is not enough to explain the navigational behavior comprehensively. *Frontiers in Behavioral Neuroscience*, *11*(154), 1–5.

Hohol, M., Cipora, K., Willmes, K., & Nuerk, H.-C. 2017b. Bringing back the balance: Domain-general processes are also important in numerical cognition. *Frontiers in Psychology*, *8*(499), 1–5.

Hohol, M., & Miłkowski, M. 2019. Cognitive artifacts for geometric reasoning. *Foundations of Science*. Online first. Retrieved from https://doi.org/10.1007/s10699-019-09603-w.

Hohol, M., Wołoszyn, K., Nuerk, H.-C., & Cipora, K. 2018. A large-scale survey on finger counting routines, their temporal stability and flexibility in educated adults. *PeerJ*, *6*(e5878), 1–23.

Hok, V., Poucet, B., Duvelle, É., Save, E., & Sargolini, F. 2016. Spatial cognition in mice and rats: Similarities and differences in brain and behavior. *Wiley Interdisciplinary Reviews: Cognitive Science*, 7(6), 406–421.

Horst, S. 2016. *Cognitive pluralism*. Cambridge, MA: The MIT Press.

Hull, C. L. 1943. *Principles of behavior: An introduction to behavior theory*. New York: Appleton-Century.

Hurley, S. 2008. The shared circuits model (SCM): How control, mirroring, and simulation can enable imitation, deliberation, and mindreading. *Behavioral and Brain Sciences*, *31*(1), 1–22.

Huttenlocher, J., Newcombe, N. S., & Vasilyeva, M. 1999. Spatial scaling in young children. *Psychological Science*, *10*(5), 393–398.

Hutto, D. D. 2019. Re-doing the math: Making enactivism add up. *Philosophical Studies*, *176*(3), 827–837.

Hwang, S., Roth, W.-M., & Kim, M. 2010. Development of mathematical concepts in children's learning geometry: A Vygotskian, body-centered approach. *Pedagogies: An International Journal*, *5*(1), 72–85.

Hyder, D. J. 2009. *The determinate world: Kant and Helmholtz on the physical meaning of geometry*. Berlin: Walter de Gruyter.

Illari, P. M., & Williamson, J. 2011. What is a mechanism? Thinking about mechanisms across the sciences. *European Journal for Philosophy of Science*, *2*(1), 119–135.

Izard, V., Pica, P., Spelke, E. S., & Dehaene, S. 2011. Flexible intuitions of Euclidean geometry in an Amazonian indigene group. *Proceedings of the National Academy of Sciences*, *108*(24), 9782–9787.

Izard, V., & Spelke, E. S. 2009. Development of sensitivity to geometry in visual forms. *Human Evolution*, *23*(3), 213–248.

Jackendoff, R. 2002. *Foundations of language: Brain, meaning, grammar, evolution*. Oxford, UK: Oxford University Press.

Jeffery, K. J., & Anderson, M. I. 2003. Dissociation of the geometric and contextual influences on place cells. *Hippocampus*, *13*(7), 868–872.

Jessen, F., Heun, R., Erb, M., Granath, D. O., Klose, U., Papassotiropoulos, A., & Grodd, W. 2000. The concreteness effect: Evidence for dual coding and context availability. *Brain and Language*, *74*(1), 103–112.

Johnson, M. 2012. *The meaning of the body*. Chicago: University of Chicago Press.

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. 2008. Development of number combination skill in the early school years: When do fingers help? *Developmental Science*, *11*(5), 662–668.

Joseph, G. G. 2011. *The crest of the peacock: Non-European roots of mathematics* (3rd ed.). Princeton, NJ: Princeton University Press.

Kahn, C. H. 2001. *Pythagoras and the Pythagoreans*. Indianapolis: Hackett Publishing.

Kamii, C., & Ewing, J. K. 1996. Basing teaching on Piaget's constructivism. *Childhood Education*, *72*(5), 260–264.

Kant, I. 2004. *Prolegomena to any future metaphysics that will be able to come forward as science with selections from the Critique of Pure Reason*. (G. Hatfield, Trans.). Cambridge, UK: Cambridge University Press.

Kapadia, R. 1974. A critical examination of Piaget-Inhelder's view on topology. *Educational Studies in Mathematics*, *5*(4), 419–424.

Kelly, D. M., Spetch, M. L., & Heth, C. D. 1998. Pigeons' (*Columba livia*) encoding of geometric and featural properties of a spatial environment. *Journal of Comparative Psychology*, *112*(3), 259–269.

Kinzler, K. D., & Spelke, E. S. 2007. Core systems in human cognition. *Progress in Brain Research*, *164*, 257–264.

Klein, E., Suchan, J., Moeller, K., Karnath, H.-O., Knops, A., Wood, G., Nuerk, H. C., & Willmes K. 2014. Considering structural connectivity in the triple code model of numerical cognition: Differential connectivity for magnitude processing and arithmetic facts. *Brain Structure and Function*, *221*(2), 979–995.

Knorr, W. R. 1974. *The evolution of the Euclidean Elements*. Dordrecht, Netherlands: Riedel Publishing.

Koedinger, K. R., & Anderson, J. R. 1990. Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science*, *14*(4), 511–550.

Koffka, K. 1936. *Principles of gestalt psychology*. London: Kagen Paul.

Kosslyn, S. M. 1980. *Image and mind*. Cambridge, MA: Harvard University Press.

Kosslyn, S. M. 1981. The medium and the message in mental imagery: A theory. *Psychological Review*, *88*(1), 46–66.

Kosslyn, S. M. 1996. *Image and brain: The resolution of the imagery debate*. New York: Bradford Books.

Kosslyn, S. M., Ball, T. M., & Reiser, B. J. 1978. Visual images preserve metric spatial information: Evidence from studies of image scanning. *Journal of Experimental Psychology*, *40*(1), 47–60.

Kosslyn, S. M., Ganis, G., & Thompson, W. L. 2003. Mental imagery: Against the nihilistic hypothesis. *Trends in Cognitive Sciences*, 7(3), 109–111.

Kosslyn, S. M., & Shwartz, S. P. 1977. A simulation of visual imagery. *Cognitive Science*, *1*(3), 265–295.

Kotarbiński, T. 1966. *Gnosiology: The scientific approach to the theory of knowledge*. Oxford, UK: Pergamon Press.

Kourtzi, Z., & Kanwisher, N. 2001. Representation of perceived object shape by the human lateral occipital complex. *Science*, *293*(5534), 1506–1509.

Kropff, E., Carmichael, J. E., Moser, M.-B., & Moser, E. I. 2015. Speed cells in the medial entorhinal cortex. *Nature*, *523*(7561), 419–424.

Króliczak, G., McAdam, T. D., Quinlan, D. J., & Culham, J. C. 2008. The human dorsal stream adapts to real actions and 3D shape processing: A functional magnetic resonance imaging study. *Journal of neurophysiology*, *100*(5), 2627–2639.

Lajoie, S. P. 2005. Extending the scaffolding metaphor. *Instructional Science*, *33*(5–6), 541–557.

Lakoff, G. 1987. *Women, fire, and dangerous things*. Chicago: University of Chicago Press.

Lakoff, G., & Johnson, M. 1980. *Metaphors we live by*. Chicago: University of Chicago Press.

Lakoff, G., & Núñez, R. E. 1997. Metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Hillsdale, NJ: Routledge.

Lakoff, G., & Núñez, R. E. 2000. *Where mathematics comes from*. New York: Basic Books.

Laland, K. N., & Brown, G. R. 2006. Niche construction, human behavior, and the adaptive-lag hypothesis. *Evolutionary Anthropology: Issues, News, and Reviews*, *15*(3), 95–104.

Landau, B. 2009. The organization and development of spatial cognition: Insights from Williams syndrome. In J. Burack (Ed.), *Handbook of intellectual disabilities and development* (pp. 60–88). Oxford, UK: Oxford University Press.

Landau, B. 2018. Learning simple spatial terms: Core and more. *Topics in Cognitive Science*. Online first. Retrieved from https://doi.org/10.1111/tops.12394.

Landau, B., Dessalegn, B., & Goldberg, A. M. 2010. Language and space: Momentary interactions. In V. Evans, & P. Chilton (Eds.), *Language, cognition and space: The state of the art and new directions* (pp. 51–77). London: Equinox.

Landau, B., & Jackendoff, R. 1993. "What" and "where" in spatial language and spatial cognition. *Behavioral and Brain Sciences*, *16*(02), 217–238.

Landau, B., Johannes, K., Skordos, D., & Papafragou, A. 2016. Containment and support: Core and complexity in spatial language learning. *Cognitive Science*, *41*(1), 748–779.

Landau, B., & Lakusta, L. 2009. Spatial representation across species: Geometry, language, and maps. *Current Opinion in Neurobiology*, *19*, 12–19.

Landy, D., Allen, C., & Zednik, C. 2014. A perceptual account of symbolic reasoning. *Frontiers in Psychology*, *5*(275), 1–10.

Larkin, J. H., & Simon, H. A. 1987. Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science*, *11*, 64–99.

Latour, B. 2008. Review essay: The Netz-works of Greek deductions. *Social Studies of Science*, *38*(3), 441–459.

Lee, K. M. 2000. Cortical areas differentially involved in multiplication and subtraction: A functional magnetic resonance imaging study and correlation with a case of selective acalculia. *Annals of Neurology*, *48*(4), 657–661.

Lee, S. A., & Spelke, E. S. 2011. Young children reorient by computing layout geometry, not by matching images of the environment. *Psychonomic Bulletin & Review*, *18*(1), 192–198.

Lee, S. A., Spelke, E. S., & Vallortigara, G. 2012. Chicks, like children, spontaneously reorient by three-dimensional environmental geometry, not by image matching. *Biology Letters*, *8*(4), 492–494.

Lee, S. A., & Vallortigara, G. 2015. Bumblebees spontaneously map location of conspecific using geometry and features. *Learning and Motivation*, *50*, 32–38.

LeFevre, J.-A. 2016. Numerical cognition: Adding it up. *Canadian Journal of Experimental Psychology*, *70*(1), 3–11.

Lehmann, D., Pascual-Marqui, R. D., Strik, W. K., & Koenig, T. 2010. Core networks for visual-concrete and abstract thought content: A brain electric microstate analysis. *NeuroImage*, *49*(1), 1073–1079.

Lenoir, T. 1979. Descartes and the geometrization of thought: The methodological background of Descartes' *Géométrie*. *Historia Mathematica*, *6*, 355–379.

Leonardis, R. 2016. The use of geometry by ancient Greek architects. In M. M. Miles (Ed.), *A companion to Greek architecture* (pp. 92–104). New York: Wiley Blackwell.

Lerman, S. 1989. Constructivism, mathematics and mathematics education. *Educational Studies in Mathematics*, *20*(2), 211–223.

Liben, L. S. 1978. Performance on Piagetian spatial tasks as a function of sex, field dependence, and training. *Merrill-Palmer Quarterly of Behavior and Development*, *24*(2), 97–110.

Liben, L. S., & Myers, L. J. 2007. Developmental changes in children's understanding of maps: What, when, and how? In J. M. Plumert, & J. P. Spencer (Eds.), *The emerging spatial mind* (pp. 193–218). Oxford, UK: Oxford University Press.

Lindemann, O., Alipour, A., & Fischer, M. H. 2011. Finger counting habits in Middle Eastern and Western individuals: An online survey. *Journal of Cross-Cultural Psychology*, *42*(4), 566–578.

Locke, J. 1999. *An essay concerning human understanding.* New York: The Pennsylvania State University.

Logothetis, N. K., & Sheinberg, D. L. 1996. Visual object recognition. *Annual Review of Neuroscience*, *19*(1), 577–621.

Lord, A. B. 1960. *The singer of tales*. Cambridge, MA: Harvard University Press.

Lovell, K. 1959. A follow-up study of some aspects of the work of Piaget and Inhelder on the child's conception of space. *British Journal of Educational Psychology*, *29*(2), 104–117.

Machery, E. 2007. Concept empiricism: A methodological critique. *Cognition*, *104*(1), 19–46.

Machery, E. 2009. *Doing without concepts*. Oxford, UK: Oxford University Press.

Machery, E. 2010. Précis of Doing without concepts. *Behavioral and Brain Sciences*, *33*(2–3), 195–206.

Machery, E. 2016. The amodal brain and the offloading hypothesis. *Psychonomic Bulletin & Review*, *23*(4), 1–6.

Mac Lane, S. 1986. *Mathematics, form and function*. New York: Springer-Verlag.

Magnani, L. 2001. *Philosophy and geometry: Theoretical and historical issues*. Dordrecht, Netherlands: Springer Science & Business Media.

Magnani, L. 2013. Thinking through drawing. *The Knowledge Engineering Review*, *28*(03), 303–326.

Maguire, E. A. 2001. The retrosplenial contribution to human navigation: A review of lesion and neuroimaging findings. *Scandinavian Journal of Psychology*, *42*(3), 225–238.

Mahon, B. Z., & Caramazza, A. 2008. A critical look at the embodied cognition hypothesis and a new proposal for grounding conceptual content. *Journal of Physiology Paris*, *102*(1-3), 59–70.

Mammarella, I. C., Giofre, D., & Caviola, S. 2017. Learning geometry: The development of geometrical concepts and the role of cognitive processes. In D. C. Geary, D. B. Berch, R. Ochsendorf, & K. M. Koepke (Eds.), *Acquisition of complex arithmetic skills and higher-order mathematics concepts* (pp. 221–246). Amsterdam: Academic Press.

Mancosu, P., Jørgensen, K. F., & Pedersen, S. A. (Eds.). 2006. *Visualization, explanation and reasoning styles in mathematics*. Dordrecht, Netherlands: Springer Science & Business Media.

Manders, K. 2008. The Euclidean diagram. In P. Mancosu (Ed.), *The philosophy of mathematical practice* (pp. 80–133). Oxford, UK: Oxford University Press.

Marghetis, T., & Núñez, R. E. 2013. The motion behind the symbols: A vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science*, *5*(2), 299–316.

Marghetis, T., Núñez, R. E., & Bergen, B. K. 2014. Doing arithmetic by hand: Hand movements during exact arithmetic reveal systematic, dynamic spatial processing. *The Quarterly Journal of Experimental Psychology*, *67*(8), 1579–1596.

Margules, J., & Gallistel, C. 1988. Heading in the rat: Determination by environmental shape. *Animal Learning & Behavior, 16*(4), 404–410.

Markman, A. B., & Dietrich, E. 1999. Whither structured representation? *Behavioral and Brain Sciences*, *22*(4), 626–627.

Marr, D. 1982. *Vision: A computational investigation into the human representation and processing of visual information*. New York: Freeman.

Marshall, P. J. 2016. Embodiment and human development. *Child Development Perspectives*, *10*(4), 245–250.

Martin, J. L. 1976a. A test with selected topological properties of Piaget's hypothesis concerning the spatial representation of the young child. *Journal for Research in Mathematics Education*, 7(1), 26–38.
Martin, J. L. 1976b. An analysis of some of Piaget's topological tasks from a mathematical point of view. *Journal for Research in Mathematics Education*, 7(1), 8–24.
Maruszewski, T. 1984. Matrix concepts, hierarchical concepts, ideal types. *Polish Psychological Bulletin*, *15*(1), 3–12.
Matyja, J. R. 2016. Embodied music cognition: Trouble ahead, trouble behind. *Frontiers in Psychology*, 7(1891), 1–3.
Mayberry, J. 1983. The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, *14*(1), 58–69.
Mayr, E. 1961. Cause and effect in biology. *Science*, *134*(3489), 1501–1506.
McGregor, A., Hayward, A. J., Pearce, J. M., & Good, M. A. 2004. Hippocampal lesions disrupt navigation based on the shape of the environment. *Behavioral Neuroscience*, *118*(5), 1011–1021.
Menary, R. (Ed.). 2010. *The extended mind*. Cambridge, MA: Bradford Books.
Menary, R. 2015. Mathematical cognition: A case of enculturation. In T. Metzinger, & J. M. Windt (Eds.), *Open MIND* (pp. 1–20). Frankfurt am Main, Germany: Johannes Gutenberg Universität Mainz.
Merleau-Ponty, M. 2002. *Phenomenology of perception*. New York: Psychology Press.
Merzbach, U. C., & Boyer, C. B. 2011. *A history of mathematics* (3rd ed.). Hoboken, NJ: John Wiley & Sons.
Mestres-Missé, A., Münte, T. F., & Rodriguez-Fornells, A. 2009. Functional neuroanatomy of contextual acquisition of concrete and abstract words. *Journal of Cognitive Neuroscience*, *21*(11), 2154–2171.
Meteyard, L., Cuadrado, S. R., Bahrami, B., & Vigliocco, G. 2012. Coming of age: A review of embodiment and the neuroscience of semantics. *Cortex*, *48*(7), 788–804.
Meyer, K., & Damasio, A. R. 2009. Convergence and divergence in a neural architecture for recognition and memory. *Trends in Neurosciences*, *32*(7), 376–382.
Migdał, P., Denkiewicz, M., Rączaszek-Leonardi, J., & Plewczyński, D. 2012. Information-sharing and aggregation models for interacting minds. *Journal of Mathematical Psychology*, *56*, 417–426.
Milford, M. J., & Wyeth, G. F. 2008. Mapping a suburb with a single camera using a biologically inspired SLAM system. *IEEE Transactions on Robotics*, *24*(5), 1038–1053.
Miłkowski, M. 2013. *Explaining the computational mind*. Cambridge, MA: The MIT Press.
Miłkowski, M. 2014. Mechanizmy reprezentacyjne i abstrakcje (Representational mechanisms and abstractions). *Przegląd Filozoficzno-Literacki*, *2*(39), 249–268.
Miłkowski, M. 2016a. Function and causal relevance of content. *New Ideas in Psychology*, *40*(Part A), 94–102.
Miłkowski, M. 2016b. Integrating cognitive (neuro)science using mechanisms. *Avant*, *6*(2), 45–67.
Miłkowski, M. 2017. Unification strategies in cognitive science. *Studies in Logic, Grammar and Rhetoric*, *48*(1), 13–33.
Miłkowski, M., Clowes, R., Rucińska, Z., Przegalińska, A., Zawidzki, T., Krueger, J., McGann, M., Afeltowicz, Ł., Wachowski, W. M., Stjernberg, F., Loughlin, V., & Hohol, M. 2018. From wide cognition to mechanisms: A silent revolution. *Frontiers in Psychology*, *9*(2393), 1–17.
Miller, G. A. 1956. The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, *63*(2), 81–97.

Miller, G. A. 2003. The cognitive revolution: A historical perspective. *Trends in Cognitive Sciences*, 7(3), 141–144.

Miller, N. 2007. *Euclid and his twentieth century rivals: Diagrams in the logic of Euclidean geometry*. Stanford, CA: CSLI Publications.

Miller, S. P., & Hudson, P. J. 2007. Using evidence-based practices to build mathematics competence related to conceptual, procedural, and declarative knowledge. *Learning Disabilities Research & Practice*, *22*(1), 47–57.

Milner, A. D., & Goodale, M. A. 2006. *The visual brain in action*. Oxford, UK: Oxford University Press.

Milner, A. D., & Goodale, M. A. 2008. Two visual systems re-viewed. *Neuropsychologia*, *46*(3), 774–785.

Minini, L., & Jeffery, K. J. 2006. Do rats use shape to solve "shape discriminations?" *Learning & Memory*, *13*(3), 287–297.

Mishkin, M., Ungerleider, L. G., & Macko, K. A. 1983. Object vision and spatial vision: Two cortical pathways. *Trends in Neurosciences*, *6*, 414–417.

Mizunami, M., Okada, R., Li, Y., & Strausfeld, N. J. 1998. Mushroom bodies of the cockroach: Activity and identities of neurons recorded in freely moving animals. *The Journal of Comparative Neurology*, *402*(4), 501–519.

Moeller, K., Willmes, K., & Klein, E. 2015. A review on functional and structural brain connectivity in numerical cognition. *Frontiers in Human Neuroscience*, *9*(227), 1–14.

Molina, M., Van de Walle, G. A., Condry, K., & Spelke, E. S. 2004. The animate-inanimate distinction in infancy: Developing sensitivity to constraints on human actions. *Journal of Cognition and Development*, *5*(4), 399–426.

Morris, J. 1973. Descartes' natural light. *Journal of the History of Philosophy*, *11*(2), 169–187.

Moser, E. I., Kropff, E., & Moser, M.-B. 2008. Place cells, grid cells, and the brain's spatial representation system. *Annual Review of Neuroscience*, *31*(1), 69–89.

Moser, E. I., Moser, M.-B., & McNaughton, B. L. 2017. Spatial representation in the hippocampal formation: A history. *Nature Publishing Group*, *20*(11), 1448–1464.

Moyer, R. S., & Landauer, T. K. 1967. Time required for judgements of numerical inequality. *Nature*, *215*(5109), 1519–1520.

Mueller, I. 1981. *Philosophy of mathematics and deductive structure in Euclid's elements*. Cambridge, MA: The MIT Press.

Mumma, J. 2009. Proofs, pictures, and Euclid. *Synthese*, *175*(2), 255–287.

Murawski, R. 2002. On the distinction proof-truth in mathematics. In P. Gärdenfors, J. Woleński, & K. Kijania-Placek (Eds.), *In the scope of logic, methodology and philosophy of science* (Vol. *1*), pp. 287–306. Dordrecht, Netherlands: Springer.

Murawski, R. 2004. Mathematical knowledge. In I. Niiniluoto, M. Sintonen, & J. Woleński (Eds.), *Handbook of epistemology* (pp. 671–606). Dordrecht, Netherlands: Springer.

Murawski, R. 2010. Truth vs. provability. Philosophical and historical remarks. In *Essays in the philosophy and history of logic and mathematics* (pp. 41–57). Amsterdam: Rodopi.

Murphy, G. 1997. Reasons to doubt the present evidence for metaphoric representation. *Cognition*, *62*(1), 99–108.

Murphy, G. L. 1996. On metaphoric representation. *Cognition*, *60*(2), 173–204.

Netz, R. 1998. Greek mathematical diagrams: Their use and their meaning. *For the Learning of Mathematics*, *18*(3), 33–39.

Netz, R. 1999a. Linguistic formulae as cognitive tools. *Pragmatics & Cognition*, 7(1), 147–176.

Netz, R. 1999b. *The shaping of deduction in Greek mathematics: A study in cognitive history.* Cambridge, UK: Cambridge University Press.

Netz, R. 2005. The limits of text in Greek mathematics. In K. Chemla (Ed.), *History of science, history of text* (pp. 161–176). Heidelberg, Germany: Springer.

Netz, R. 2009. Imagination and layered ontology in Greek mathematics. *Configurations*, *17*(1), 19–50.

Netz, R., & Noel, W. 2009. *The Archimedes codex*. Cambridge, MA: Da Capo Press.

Nevins, A. J. 1975. Plane geometry theorem proving using forward chaining. *Artificial Intelligence*, *6*(1), 1–23.

Newcombe, N. S. 1989. The development of spatial perspective taking. In H. W. Reese (Ed.), *Advances in child development and behavior* (pp. 203–247). New York: Academic Press.

Newell, A., Shaw, J. C., & Simon, H. A. 1959. Report on a general problem solving program. In *Proceedings of the International Conference on Information Processing* (pp. 256–264). Paris: UNESCO House.

Newell, A., & Simon, H. A. 1956. The logic theory machine: A complex information processing system. *IEEE Transactions on Information Theory*, *2*(3), 61–79.

Newell, A., & Simon, H. A. 1972. *Human problem solving*. Englewood Cliffs, NJ: Prentice Hall.

Newell, A., & Simon, H. A. 1976. Computer science as empirical inquiry: Symbols and search. *Communications of the ACM*, *19*(3), 113–126.

Niedenthal, P. M., Winkielman, P., Mondillon, L., & Vermeulen, N. 2009. Embodiment of emotion concepts. *Journal of Personality and Social Psychology*, *96*(6), 1120–1136.

Nigel, J. T. T. 2018. Mental imagery. In E. Zalta (Ed.), *Stanford Encyclopedia of Philosophy (Spring Edition)*. Stanford. Retrieved from https://plato.stanford.edu/archives/spr2018/entries/mental-imagery/.

Ninio, A. 1979. Piaget's theory of space perception in infancy. *Cognition*, 7(2), 125–144.

Noë, A. 2009. *Out of our heads*. New York: Hill and Wang.

Noël, M.-P. 2005. Finger gnosia: A predictor of numerical abilities in children? *Child Neuropsychology*, *11*(5), 413–430.

Noppeney, U., & Price, C. J. 2004. Retrieval of abstract semantics. *NeuroImage*, *22*(1), 164–170.

Norman, D. A. 1991. Cognitive artifacts. In J. M. Carroll (Ed.), *Designing interaction: Psychology at the human-computer interface* (pp. 17–38). Cambridge, UK: Cambridge University Press.

Núñez, R. E. 2017a. Is there really an evolved capacity for number? *Trends in Cognitive Sciences*, *21*(6), 409–424.

Núñez, R. E. 2017b. Number—Biological enculturation beyond natural selection. *Trends in Cognitive Sciences*, *21*(6), 404–405.

Núñez, R., Allen, M., Gao, R., Rigoli, C. M., Relaford-Doyle, J., & Semenuks, A. 2019. What happened to cognitive science? *Nature Human Behaviour*, *3*(8), 782–791.

Núñez, R. E., & Lakoff, G. 1998. What did Weierstrass really define? The cognitive structure of natural and ∈–δ continuity. *Mathematical Cognition*, *4*(2), 85–101.

OECD. 2014. *PISA 2012 results: What students know and can do (volume I, revised edition): Student performance in mathematics, reading and science*. OECD, Paris.

O'Grady, P. F. 2002. *Thales of Miletus*. New York: Taylor & Francis.

O'Keefe, J., & Dostrovsky, J. 1971. The hippocampus as a spatial map: Preliminary evidence from unit activity in the freely-moving rat. *Brain Research*, *34*(1), 171–175.

O'Keefe, J., & Nadel, L. 1978. *The hippocampus as a cognitive map*. Oxford, UK: Oxford University Press.

O'Leary, M. 2010. *Revolutions of geometry*. Hoboken, NJ: John Wiley & Sons.

O'Meara, D. 2017. Mathematics and the sciences. In P. D'Hoine, & M. Martijn (Eds.), *All from one: A guide to Proclus* (pp. 167–182). Oxford, UK: Oxford University Press.

Oppel, J. J. 1855. Uber geometrisch-optische Täuschungen (On geometrical-optical illusions). *Jahresbericht Des Physikalischen Vereins Zu Frankfurt Am Main*, *1854–1855*, 37–47.

Overmann, K. A. 2014. Finger-counting in the Upper Palaeolithic. *Rock Art Research*, *31*(1), 63–80.

Page, E. I. 1959. Haptic perception: A consideration of one of the investigations of Piaget and Inhelder. *Educational Review*, *11*(2), 115–124.

Paivio, A. 1971. *Imagery and verbal processes*. New York: Psychology Press.

Paivio, A. 1986. *Mental representations: A dual coding approach*. Oxford, UK: Oxford University Press.

Paivio, A. 1991. Dual coding theory: Retrospect and current status. *Canadian Journal of Psychology*, *45*(3), 255–287.

Panza, M. 2012. The twofold role of diagrams in Euclid's plane geometry. *Synthese*, *186*(1), 55–102.

Papagno, C., Fogliata, A., Catricalà, E., & Miniussi, C. 2009. The lexical processing of abstract and concrete nouns. *Brain Research*, *1263*, 78–86.

Papesh, M. H. 2015. Just out of reach: On the reliability of the action-sentence compatibility effect. *Journal of Experimental Psychology: General*, *144*(6), e116–e141.

Parry, M. 1971. *The making of Homeric verse*. Oxford, UK: Clarendon Press.

Parsons, C. 1980. Mathematical intuition. *Proceedings of the Aristotelian Society*, *80*(1), 145–168.

Pasch, M. 1912. *Vorlesungen über die neuere Geometrie*. Leipzig, Germany: Teubner.

Patro, K., & Haman, M. 2012. The spatial–numerical congruity effect in preschoolers. *Journal of Experimental Child Psychology*, *111*(3), 534–542.

Pearson, J., & Kosslyn, S. M. 2015. The heterogeneity of mental representation: Ending the imagery debate. *Proceedings of the National Academy of Sciences*, *112*(33), 10089–10092.

Pecher, D., Zeelenberg, R., & Barsalou, L. W. 2003. Verifying different-modality properties for concepts produces switching costs. *Psychological Science*, *14*(2), 119–124.

Pecher, D., & Zwaan, R. A. 2005. *Grounding cognition: The role of perception and action in memory, language, and thinking*. Cambridge, UK: Cambridge University Press.

Peel, E. A. 1959. Experimental examination of some of Piaget's schemata concerning children's perception and thinking, and a discussion of their educational significance. *British Journal of Educational Psychology*, *29*(2), 89–103.

Peirce, C. S. 1976. Lowell lectures. Lecture 2. In C. Eisele (Ed.), *The new elements of mathematics, Vol. III/1* (pp. 405–430). The Hague, Netherlands: Mouton Publishers.

Penner-Wilger, M., & Anderson, M. L. 2013. The relation between finger gnosis and mathematical ability: Why redeployment of neural circuits best explains the finding. *Frontiers in Psychology*, *4*(877), 1–9.

Penrose, R. 2004. *The road to reality: A complete guide to the laws of the universe*. New York: Alfred A. Knopf.

Penrose, R. 2018. Experiences with Stephen Hawking. *The Oxford Mathematics Newsletter*, Spring, 4–5.

Perry, M. 1930. Studies in the epic technique of oral verse-making: I. Homer and homeric style. *Harvard Studies in Classical Philology*, *41*, 73–147.

Persichetti, A. S., & Dilks, D. D. 2016. Perceived egocentric distance sensitivity and invariance across scene-selective cortex. *Cortex*, *77*, 155–163.

Pesenti, M., Thioux, M., Seron, X., & De Volder, A. 2000. Neuroanatomical substrates of Arabic number processing, numerical comparison, and simple addition: A PET Study. *Journal of Cognitive Neuroscience*, *12*(3), 461–479.

Pfeiffer, K., & Homberg, U. 2014. Organization and functional roles of the central complex in the insect brain. *Annual Review of Entomology*, *59*(1), 165–184.

Piaget, J. 1926. *Language and thought of the child*. New York: Harcourt.

Piaget, J., & Inhelder, B. 1967. *The child's conception of space*. New York: W.W. Norton & Co.

Piaget, J., Inhelder, B., & Szeminska, A. 1960. *The child's conception of geometry*. New York: W. W. Norton Company.

Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. 2004. Tuning curves for approximate numerosity in the human intraparietal sulcus. *Neuron*, *44*(3), 547–555.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. 2004. Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499–503.

Piccinini, G. 2007. Computing mechanisms. *Philosophy of Science*, *74*(4), 501–526.

Piccinini, G., & Craver, C. F. 2011. Integrating psychology and neuroscience: Functional analyses as mechanism sketches. *Synthese*, *183*(3), 283–311.

Pinker, S. 2009. *How the mind works*. New York: W. W. Norton & Company.

Plato. 2009a. *Meno*. (B. Jowett, Trans.). Cambridge, MA: The MIT: The Internet Classics Archive. Retrieved from http://classics.mit.edu/Plato/meno.html.

Plato. 2009b. *The Republic*. (B. Jowett, Trans.). Cambridge, MA: The MIT: The Internet Classics Archive.

Pogonowski, J. 2017. On conceptual metaphors in mathematics. *Annales Universitatis Paedagogicae Cracoviensis Studia ad Didacticam Mathematicae Pertinentia*, *9*, 85–98.

Poincaré, H. 1905. *Science and hypothesis*. (W. J. Greenstreet, Trans.). New York: Walter Scott.

Poincaré, H. 1929. *The foundations of science*. (B. Halsted, Trans.). New York: The Science Press.

Ponticorvo, M., & Miglino, O. 2009. Encoding geometric and non-geometric information: A study with evolved agents. *Animal Cognition*, *13*(1), 157–174.

Poucet, B., Chapuis, N., Durup, M., & Thinus-Blanc, C. 1986. A study of exploratory behavior as an index of spatial knowledge in hamsters. *Animal Learning & Behavior*, *14*(1), 93–100.

Prinz, J. J. 2004. *Furnishing the mind: Concepts and their perceptual basis*. Cambridge, MA: The MIT Press.

Prinz, J. J. 2006. Is the mind really modular? In R. J. Stainton (Ed.), *Contemporary debates in cognitive science* (pp. 22–36). Malden, MA: Blackwell.

Prinz, J. J. 2010. Can concept empiricism forestall eliminativism? *Mind & Language*, *25*(5), 612–621.

Proclus. 1970. *A commentary on the first book of Euclid's elements*. (G. R. Morrow, Trans.). Princeton, NJ: Princeton University Press.

Pulvermüller, F. 2002. *The neuroscience of language: On brain circuits of words and serial order*. Cambridge, UK: Cambridge University Press.

Pulvermüller, F., Hauk, O., Nikulin, V. V., & Ilmoniemi, R. J. 2005. Functional links between motor and language systems. *European Journal of Neuroscience*, *21*(3), 793–797.

Putnam, H. 1967. Psychological predicates. In W. H. Capitan, & D. D. Merrill (Eds.), *Art, mind, and religion* (pp. 37–48). Pittsburgh: University of Pittsburgh Press.

Pylyshyn, Z. W. 1973. What the mind's eye tells the mind's brain: A critique of mental imagery. *Psychological Bulletin*, *80*(1), 1–24.

Pylyshyn, Z. W. 1981. The imagery debate: Analogue media versus tacit knowledge. *Psychological Review*, *88*(1), 16–45.

Pylyshyn, Z. W. 1984. *Computation and cognition: Toward a foundation for cognitive science*. Cambridge, MA: The MIT Press.

Rączaszek-Leonardi, J. 2009. Symbols as constraints: The structuring role of dynamics and self-organization in natural language. *Pragmatics & Cognition*, *17*(3), 653–676.

Rączaszek-Leonardi, J., & Cowley, S. J. 2012. The evolution of language as controlled collectivity. *Interaction Studies*, *13*(1), 1–16.

Ramsey, W. M. 2007. *Representation reconsidered*. Cambridge, UK: Cambridge University Press.

Ravassard, P., Kees, A., Willers, B., Ho, D., Aharoni, D., Cushman, J., Aghajan Z. M., & Mehta, M. R. 2013. Multisensory control of hippocampal spatiotemporal selectivity. *Science*, *340*(6138), 1342–1346.

Reichenbach, H. 1938. *Experience and prediction: An analysis of the foundations and the structure of knowledge*. Chicago: Chicago University Press.

Reynolds, R. E., & Ortony, A. 1980. Some issues in the measurement of children's comprehension of metaphorical language. *Child Development*, *51*(4), 1110–1119.

Rosch, E. 1978. Principles of categorization. In E. Rosch, & B. B. Lloyd (Eds.), *Cognition and categorization* (pp. 28–49). Hillsdale, NJ: Lawrence Erlbaum.

Rota, G.-C. 1997. The phenomenology of mathematical beauty. *Synthese*, *111*(2), 171–182.

Roth, W.-M. 2011. *Geometry as objective science in elementary school classrooms*. New York: Routledge.

Roth, W.-M., & Lee, Y.-J. 2016. "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research*, 77(2), 186–232.

Russell, B. 2009. *Autobiography*. New York: Routledge.

Russo, L. 2004. *The forgotten revolution*. Berlin: Springer.

Sabsevitz, D. S., Medler, D. A., Seidenberg, M., & Binder, J. R. 2005. Modulation of the semantic system by word imageability. *NeuroImage*, *27*(1), 188–200.

Saito, K., & Sidoli, N. 2012. Diagrams and arguments in ancient Greek mathematics: Lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions. In K. Chemla (Ed.), *The history of mathematical proof in ancient traditions* (pp. 135–162). Cambridge, UK: Cambridge University Press.

Samhita, L., & Gross, H. J. 2013. The "Clever Hans Phenomenon" revisited. *Communicative & Integrative Biology*, *6*(6), e27122–e27124.

Samuels, R. 2004. Innateness in cognitive science. *Trends in Cognitive Sciences*, *8*(3), 136–141.

Santiago, J., Lupáñez, J., Pérez, E., & Funes, M. J. 2007. Time (also) flies from left to right. *Psychonomic Bulletin & Review*, *14*(3), 512–516.

Sarton, G. 1959. *Ancient science and modern civilization*. New York: Harper & Brothers.

Saxe, G. B. 2014. *Cultural development of mathematical ideas: Papua New Guinea studies*. Cambridge, UK: Cambridge University Press.

Schmithorst, V. J., & Brown, R. D. 2004. Empirical validation of the triple-code model of numerical processing for complex math operations using functional MRI and group Independent Component Analysis of the mental addition and subtraction of fractions. *NeuroImage*, *22*, 1414–1420.

Scriba, C. J., & Schreiber, P. 2015. *5000 years of geometry: Mathematics in history and culture*. Basel, Switzerland: Birkhäuser.

Seidenberg, A. 1959. Peg and cord in ancient Greek geometry. *Scripta Mathematica*, *24*(2), 107–122.

Seidenberg, A. 1961. The ritual origin of geometry. *Archive for History of Exact Sciences*, *1*(5), 488–527.

Sella, F., Sader, E., Lolliot, S., & Cohen Kadosh, R. 2016. Basic and advanced numerical performances relate to mathematical expertise but are fully mediated by visuospatial skills. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(9), 1458–1472.

Shaki, S., & Fischer, M. H. 2018. Deconstructing spatial-numerical associations. *Cognition*, *175*, 109–113.

Shaki, S., Fischer, M. H., & Petrusic, W. M. 2009. Reading habits for both words and numbers contribute to the SNARC effect. *Psychonomic Bulletin & Review*, *16*(2), 328–331.

Shallice, T., & Cooper, R. P. 2013. Is there a semantic system for abstract words? *Frontiers in Human Neuroscience*, 7(175), 1–10.

Shannon, C. E., & McCarthy, J. (Eds.). 1956. *Automata studies*. Princeton, NJ: Princeton University Press.

Shepard, R. N. 1975. Form, formation, and transformation of internal representations. In R. Solso (Ed.), *Information processing and cognition the Loyola symposium* (pp. 87–122). Hillsdale, NJ: Lawrence Erlbaum.

Shepard, R. N., & Metzler, J. 1971. Mental rotation of three-dimensional objects. *Science*, *171*(3972), 701–703.

Sherry, D. 2008. The role of diagrams in mathematical arguments. *Foundations of Science*, *14*(1-2), 59–74.

Sherry, D. F., Jacobs, L. F., & Gaulin, S. J. C. 1992. Spatial memory and adaptive specialization of the hippocampus. *Trends in Neurosciences*, *15*(8), 298–303.

Sheynikhovich, D., Chavarriaga, R., Strösslin, T., Arleo, A., & Gerstner, W. 2009. Is there a geometric module for spatial orientation? Insights from a rodent navigation model. *Psychological Review*, *116*(3), 540–566.

Shutts, K., Örnkloo, H., von Hofsten, C., Keen, R., & Spelke, E. S. 2009. Young children's representations of spatial and functional relations between objects. *Child Development*, *80*(6), 1612–1627.

Simmons, W. K., Ramjee, V., Beauchamp, M. S., McRae, K., Martin, A., & Barsalou, L. W. 2007. A common neural substrate for perceiving and knowing about color. *Neuropsychologia*, *45*(12), 2802–2810.

Simon, H. A. 1978. On the forms of mental representation. In W. Savage (Ed.), *Perception and cognition issues in the foundations of psychology* (pp. 3–18). Minneapolis: University of Minnesota Press.

Sinclair, N., Moss, J., Hawes, Z., & Stephenson, C. 2018. Learning through and from drawing in early years geometry. In K. S. Mix, & M. T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought* (pp. 229–252). Cham, Switzerland: Springer.

Sinclair, N., & Schiralli, M. 2003. A constructive response to "Where mathematics comes from". *Educational Studies in Mathematics*, *52*(1), 79–91.

Skrzypczyński, P. 2002. Simultaneous localization and mapping: A feature-based probabilistic approach. *International Journal of Applied Mathematics and Computer Science*, *19*(4), 575–588.

Sloman, A. 1975. Afterthoughts on analogical representations. In R. C. Schank, & B. L. Nash-Webber (Eds.), *Theoretical issues in natural language processing*. Arlington, VA: Tinlap Press, 164–168.

Sloman, A. 2002. Diagrams in mind? In M. Anderson, B. Meyer, & P. Olivier (Eds.), *Diagrammatic representation and reasoning* (pp. 7–28). London: Springer-Verlag.

Smith, E. E., Osherson, D. N., Rips, L. J., & Keane, M. 1988. Combining prototypes: A selective modification model. *Cognitive Science*, *12*(4), 485–527.

Smith, E. A., Mulder, M. B., & Hill, K. 2001. Controversies in the evolutionary social sciences: A guide for the perplexed. *Trends in Ecology & Evolution*, *16*(3), 128–135.

Solstad, T., Boccara, C. N., Kropff, E., Moser, M.-B., & Moser, E. I. 2008. Representation of geometric borders in the entorhinal cortex. *Science*, *322*(5909), 1865–1868.

Sovrano, V. A., Bisazza, A., & Vallortigara, G. 2002. Modularity and spatial reorientation in a simple mind: Encoding of geometric and nongeometric properties of a spatial environment by fish. *Cognition*, *85*(2), B51–B59.

Soylu, F., Lester, F. K., & Newman, S. D. 2018. You can count on your fingers: The role of fingers in early mathematical development. *Journal of Numerical Cognition*, *4*(1), 107–135.

Spelke, E. S. 2000. Core knowledge. *American Psychologist*, *55*(11), 1233–1243.

Spelke, E. S. 2013. What makes us smart? Core knowledge and natural language. In D. Gentner, & S. Goldin-Meadow (Eds.), *Language in mind Advances in the study of language and thought* (pp. 277–311). Cambridge, MA: The MIT Press.

Spelke, E. S., Breinlinger, K., Macomber, J., & Jacobson, K. 1992. Origins of knowledge. *Psychological Review*, *99*(4), 605–632.

Spelke, E. S., & Kinzler, K. D. 2007. Core knowledge. *Developmental Science*, *10*(1), 89–96.

Spelke, E. S., & Lee, S. A. 2012. Core systems of geometry in animal minds. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *367*(1603), 2784–2793.

Spelke, E. S., Lee, S. A., & Izard, V. 2010. Beyond core knowledge: Natural geometry. *Cognitive Science*, *34*(5), 863–884.

Sperber, D. 2001. In defense of massive modularity. In E. Dupoux (Ed.), *Language, brain and cognitive development Essays in honor of Jacques Mehler* (pp. 47–57). Cambridge, MA: The MIT Press.

Spokes, A. C., & Spelke, E. S. 2017. The cradle of social knowledge: Infants' reasoning about caregiving and affiliation. *Cognition*, *159*, 102–116.

Sterelny, K. 2010. Minds: Extended or scaffolded? *Phenomenology and the Cognitive Sciences*, *9*(4), 465–481.

Stürzl, W., Cheung, A., Cheng, K., & Zeil, J. 2008. The information content of panoramic images I: The rotational errors and the similarity of views in rectangular experimental arenas. *Journal of Experimental Psychology: Animal Behavior Processes*, *34*(1), 1–14.

Sutton, J. E., & Newcombe, N. S. 2014. The hippocampus is not a geometric module: Processing environment geometry during reorientation. *Frontiers in Human Neuroscience*, *8*(596), 1–6.

Szabó, A. 1978. *The beginnings of Greek mathematics*. Dordrecht, Netherlands: Reidel Publishing.

Tarski, A. 1899/1950. What is elementary geometry? In L. Henkin, P. Suppes, & A. Tarski (Eds.), The axiomatic method. With special reference to geometry and physics. *Proceedings of an International Symposium held at the University of California* (pp. 16–29), Berkeley, CA: University of California. December 26, 1957–January 4, 1958.

Taube, J. S., Muller, R. U., & Ranck, J. B. 1990. Head-direction cells recorded from the postsubiculum in freely moving rats. I. Description and quantitative analysis. *Journal of Neuroscience*, *10*(2), 420–435.

Thinus-Blanc, C., Chabanne, V., Tomassi, L., Peruch, P., & Vauclair, J. 2010. The encoding of geometry in various vertebrate species. In F. L. Dolins, & R. W. Mitchell (Eds.), *Spatial cognition, spatial perception* (pp. 99–116). Cambridge, UK: Cambridge University Press.

Thomas, R. 1992. *Literacy and orality in ancient Greece*. Cambridge, UK: Cambridge University Press.

Thomson, E., & Piccinini, G. 2018. Neural representations observed. *Minds and Machines*, *28*(1), 191–235.

Thrun, S., Burgard, W., & Fox, D. 2005. *Probabilistic robotics*. Cambridge, MA: The MIT Press.

Tinbergen, N. 1963. On aims and methods of ethology. *Zeitschrift Für Tierpsychologie*, *20*(3), 410–433.

Tolman, E. C. 1932. *Purposive behavior in animals and man*. Berkeley, CA: University of California Press.

Tolman, E. C. 1948. Cognitive maps in rats and men. *Psychological Review*, *55*(4), 189–208.

Tomasello, M. 2009. *The cultural origins of human cognition*. Cambridge, MA: Harvard University Press.

Tomassi, L., & Thinus-Blanc, C. 2004. Generalization in place learning and geometry knowledge in rats. *Learning & Memory*, *11*(2), 153–161.

Tomassi, L., Vallortigara, G., & Zanforlin, M. 1997. Young chickens learn to localize the centre of a spatial environment. *Journal of Comparative Physiology A: Sensory, Neural, and Behavioral Physiology*, *180*(5), 567–572.

Trudeau, R. J. 2009. *The non-Euclidean revolution*. Boston: Birkhauser.

Tschentscher, N., Hauk, O., Fischer, M. H., & Pulvermüller, F. 2012. You can count on the motor cortex: Finger counting habits modulate motor cortex activation evoked by numbers. *NeuroImage*, *59*(4), 3139–3148.

Tversky, B. 1993. Cognitive maps, cognitive collages, and spatial mental models. In *European conference on spatial information theory* (pp. 14–24). Heidelberg, Germany: Springer.

Tversky, B. 2009. Spatial cognition: Embodied and situated. In P. Robbins, & M. Aydede (Eds.), *The Cambridge handbook of situated cognition* (pp. 201–216). Cambridge, UK: Cambridge University Press.

Twyman, A. D., Newcombe, N. S., & Gould, T. J. 2009. Of mice (*Mus musculus*) and toddlers (*Homo sapiens*): Evidence for species-general spatial reorientation. *Journal of Comparative Psychology*, *123*(3), 342–345.

Tylén, K., Fusaroli, R., Bjørndahl, J. S., Rączaszek-Leonardi, J., Østergaard, S., & Stjernfelt, F. 2014. Diagrammatic reasoning: Abstraction, interaction, and insight. *Pragmatics & Cognition*, *22*(2), 264–283.

Tymoczko, T. 1979. The four-color problem and its philosophical significance. *The Journal of Philosophy*, *76*(2), 57–83.

Tzelgov, J., Meyer, J., & Henik, A. 1992. Automatic and intentional processing of numerical information. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *18*(1), 166–179.

Usiskin, Z. 1982. *Van Hiele levels and achievement in secondary school geometry*. Chicago: University of Chicago.

Vallortigara, G. 2012. Core knowledge of object, number, and geometry: A comparative and neural approach. *Cognitive Neuropsychology*, *29*(1-2), 213–236.

Vallortigara, G. 2018. Comparative cognition of number and space: The case of geometry and of the mental number line. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *373*(1740), 20170120–20170128.

Vallortigara, G., Zanforlin, M., & Pasti, G. 1990. Geometric modules in animals' spatial representations: A test with chicks (*Gallus gallus domesticus*). *Journal of Comparative Psychology*, *104*(3), 248–254.

van Elk, M., Slors, M., & Bekkering, H. 2010. Embodied language comprehension requires an enactivist paradigm of cognition. *Frontiers in Psychology*, *1*(234), 1–9 .

van Hiele, P. M. 1986. *Structure and insight*. Orlando, FL: Academic Press.

van Hiele-Geldof, D., & van Hiele, P. M. 1984. *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. (D. Fuys, D. Geddes, & R. Tischler, Eds.). Brooklyn, NY: Brooklyn College.

Varela, F. J., Thompson, E., & Rosch, E. 1991. *The embodied mind*. Cambridge, MA: The MIT Press.

Vargas, J. P., López, J. C., Salas, C., & Thinus-Blanc, C. 2004a. Encoding of geometric and featural spatial information by goldfish (*Carassius auratus*). *Journal of Comparative Psychology*, *118*(2), 206–216.

Vargas, J. P., Petruso, E. J., & Bingman, V. P. 2004b. Hippocampal formation is required for geometric navigation in pigeons. *European Journal of Neuroscience*, *20*(7), 1937–1944.

Vigliocco, G., Vinson, D. P., Druks, J., Barber, H., & Cappa, S. F. 2011. Nouns and verbs in the brain: A review of behavioural, electrophysiological, neuropsychological and imaging studies. *Neuroscience & Biobehavioral Reviews*, *35*(3), 407–426.

Vlastos, G. 1995. Zeno of Sidon as a critic of Euclid. In D. W. Graham (Ed.), *Studies in Greek philosophy*. Vol. II: Socrates Plato and their tradition (pp. 315–324). Princeton: Princeton University Press.

von Helmholtz, H. 1962. *Treatise on physiological optics*. New York: Optical Society of America. Retrieved from http://poseidon.sunyopt.edu/BackusLab/Helmholtz/.

von Helmholtz, H. 1977a. On the facts underlying geometry. In P. Hertz, & M. Schlick (Eds.), *Epistemological writings* (pp. 39–71). Dordrecht, Netherlands: Riedel.

von Helmholtz, H. 1977b. On the origin and significance of the axioms of geometry. In P. Hertz, & M. Schlick (Eds.), *Epistemological writings* (pp. 1–38). Dordrecht, Netherlands: Riedel.

Voorhees, B. 2004. Embodied mathematics: Comments on Lakoff & Núñez. *Journal of Consciousness Studies*, *11*(9), 83–88.

Vygotsky, L. S. 1986. *Thought and language* (Newly revised and edited by Alex Kozulin). Cambridge, MA: The MIT Press.

Wang, J., Conder, J. A., Blitzer, D. N., & Shinkareva, S. V. 2010. Neural representation of abstract and concrete concepts: A meta-analysis of neuroimaging studies. *Human Brain Mapping*, *31*(10), 1459–1468.

Wasner, M., Moeller, K., Fischer, M. H., & Nuerk, H.-C. 2014. Aspects of situated cognition in embodied numerosity: The case of finger counting. *Cognitive Processing*, *15*(3), 317–328.

Wattenmaker, W. D., & Shoben, E. J. 1987. Context and the recallability of concrete and abstract sentences. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *13*(1), 140–150.

Webb, B. 2012. Cognition in insects. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, *367*(1603), 2715–2722.

Wedberg, A. 1955. *Plato's philosophy of mathematics*. Stockholm: Almqvist & Wiksell.

Whorf, B. L. 1956. *Language, thought, and reality: Selected writings of Benjamin Lee Whorf* (J. B. Carroll, Ed.). Cambridge, MA: The MIT Press.

Wiemer-Hastings, K., & Xu, X. 2005. Content differences for abstract and concrete concepts. *Cognitive Science*, *29*(5), 719–736.

Williams, G. C. 1966. *Adaptation and natural selection*. Princeton, NJ: Princeton University Press.

Wilson, M. 2002. Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636.

Wilson, R. A. 2004. *Boundaries of the mind*. Cambridge, UK: Cambridge University Press.

Wilson-Mendenhall, C. D., Simmons, W. K., Martin, A., & Barsalou, L. W. 2013. Contextual processing of abstract concepts reveals neural representations of nonlinguistic semantic content. *Journal of Cognitive Neuroscience*, *25*(6), 920–935.

Winkielman, P., Coulson, S., & Niedenthal, P. M. 2018. Dynamic grounding of emotion concepts. *Philosophical Transactions of the Royal Society B: Biological Sciences*, *373*(1752), 20170127–20170135.

Winkielman, P., Niedenthal, P. M., & Oberman, L. M. 2008. The embodied emotional mind. In G. R. Semin, & E. R. Smith (Eds.), *Embodied grounding: Social, cognitive, affective, and neuroscientific approaches* (pp. 263–288). Cambridge, UK: Cambridge University Press.

Winkielman, P., Niedenthal, P. M., Wielgosz, J., Eelen, J., & Kavanagh, L. 2015. Embodiment of cognition and emotion. In M. Mikulincer, P. R. Shaver, E. Borgida, & J. A. Bargh (Eds.), *APA handbook of personality and social psychology, Vol. 1. Attitudes and social cognition* (pp. 151–175). Washington, DC: American Psychological Association.

Winkler-Rhoades, N., Carey, S. C., & Spelke, E. S. 2013. Two-year-old children interpret abstract, purely geometric maps. *Developmental Science*, *16*(3), 365–376.

Winner, E., Rosenstiel, A. K., & Gardner, H. 1976. The development of metaphoric understanding. *Developmental Psychology*, *12*(4), 289–297.

Wirszup, I. 1976. Breakthroughs in the psychology of learning and teaching geometry. In J. L. Martin, & D. A. Bradbard (Eds.), *Space and geometry: Papers from a research workshops* (pp. 75–97). Columbus, OH: Ohio State University.

Woleński, J. 1987. Reism and Leśniewski's ontology. *History and Philosophy of Logic*, 7(2), 167–176.

Woleński, J. 1989. *Logic and philosophy in the Lvov-Warsaw school*. Dordrecht, Netherlands: Springer.

Wołoszyn, K., & Hohol, M. 2017. Commentary: The poverty of embodied cognition. *Frontiers in Psychology*, *8*(845), 1–3.

Wood, D., Bruner, J. S., & Ross, G. 1976. The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17*(2), 89–100.

Wood, G., Nuerk, H.-C., & Willmes, K. 2006. Neural representations of two-digit numbers: A parametric fMRI study. *NeuroImage*, *29*(2), 358–367.

Wood, G., Willmes, K., & Nuerk, H.-C. 2008. On the cognitive link between space and number: A meta-analysis of the SNARC effect. *Psychology Science Quarterly*, *50*(4), 489–525.

Wray, M. K., Klein, B. A., Mattila, H. R., & Seeley, T. D. 2008. Honeybees do not reject dances for "implausible" locations: Reconsidering the evidence for cognitive maps in insects. *Animal Behaviour*, *76*(2), 261–269.

Wu, L.-L., & Barsalou, L. W. 2009. Perceptual simulation in conceptual combination: Evidence from property generation. *Acta Psychologica*, *132*(2), 173–189.

Wundt, W. M. 1898. *Die geometrisch-optischen Täuschungen*. Leipzig, Germany: Teubner.

Wystrach, A., & Beugnon, G. 2009. Ants learn geometry and features. *Current Biology*, *19*(1), 61–66.

Zago, L., Pesenti, M., Mellet, E., Crivello, F., Mazoyer, B., & Tzourio-Mazoyer, N. 2001. Neural correlates of simple and complex mental calculation. *NeuroImage*, *13*(2), 314–327.

Zeki, S., Romaya, J. P., Benincasa, D. M. T., & Atiyah, M. F. 2014. The experience of mathematical beauty and its neural correlates. *Frontiers in Human Neuroscience*, *8*(68), 1–12.

Zoccolan, D., Oertelt, N., DiCarlo, J. J., & Cox, D. D. 2009. A rodent model for the study of invariant visual object recognition. *Proceedings of the National Academy of Sciences*, *106*(21), 8748–8875.

Zohar-Shai, B., Tzelgov, J., Karni, A., & Rubinsten, O. 2017. It does exist! A left-to-right spatial–numerical association of response codes (SNARC) effect among native Hebrew speakers. *Journal of Experimental Psychology: Human Perception and Performance*, *43*(4), 719–728.

Zwaan, R. A., & Kaschak, M. P. 2009. Language in the brain, body, and world. In P. Robbins, & M. Aydede (Eds.), *The Cambridge handbook of situated cognition* (pp. 368–381). Cambridge, UK: Cambridge University Press.

## Notes

1. This section does not aspire to be a complete historical reconstruction of the origins of geometry. What is more, I am aware that its content is simplified and the achievements of several Ancient mathematicians, such as Eudoxus of Cnidus, are omitted here. Readers interested in the beginnings and further development of geometry should refer to professional textbooks, historical monographs and collections of essays (see, e.g., De Risi, 2015; Goodman, 2016; Heller, 2019; Merzbach & Boyer, 2011; O'Leary, 2010; Scriba & Schreiber, 2015).
2. Some traditional editions of *Elements* were enriched by two additional "apocryphal" books on regular solids. Today, we know that Euclid was not their author. The authorship of the XIV book is attributed to Hypiscles of Alexandria, and the XV book is, at least partly, a work by Isidore of Miletus.
3. Noteworthy, the purpose of so-called definitions introduced by Euclid is to facilitate the grasp of the meaning of geometric concepts, but they are not definitions in the strict logical sense. Some of them are circular—they do not satisfy the requirement that a definiens should be better known than definiendum (Merzbach & Boyer, 2011, p. 95; Russo, 2004, pp. 320–237). For this and other reasons (see the note 4), I said earlier that Euclid's method is *modeled* on the Aristotelian idea.
4. It should be noted that Euclid's distinction between postulates and common notions cannot be directly identified with an Aristotelian division of principles into axioms and postulates. According to Merzbach and Boyer (2011), "we do not know whether Euclid distinguished between two types of assumptions. Surviving manuscripts are not in agreement here, and in some cases, the ten assumptions appear together in a single category. Modern mathematicians see no essential difference between an axiom and a postulate" (p. 95). But how do Euclidean postulates and common notions concepts differ? In *A Commentary on the First Book of Euclid's Elements*, Proclus

(1970) said that although both kinds of initial statements are self evident and easy to grasp, the former take the form of effortless construction problems or tasks (i.e., "to draw a straight-line from any point to any point"), whereas the later are obvious assertions (i.e., "Things equal to the same thing are also equal to one another"). As Proclus says, "a postulate prescribes that we construct or provide some simple or easily grasped object for the exhibition of a character, while an axiom (common notion—M.H.) asserts some inherent attribute that is known at once to one's auditors (…). So a postulate has the same general character as an axiom but differs from it in the manner described" (Proclus, 1970, p. 181).

5. As the scope of the book is limited solely to Euclidean geometry, every reader interested in the non-Euclidean geometries should reach for other elaborations. The books by Greenberg (1993) and Trudeau (2009) are useful introductions.
6. In opposition, *analytic* means that a predicate concept of a proposition is contained in its subject concept (e.g. "all bachelors are unmarried"); *a posteriori* means, in turn, that the justification of proposition depends upon empirical experience ("all bachelors are sad").
7. Note, however, that Helmholtz—as Biagioli (2016) pointed out—"made it clear that geometrical assumptions cannot be tested directly. Such a test must be indirect because of the origin of geometrical axioms. Though Helmholtz maintained that geometrical axioms have empirical origins, he emphasized the role of cognitive functions and inferences in the formation of geometrical notions. Geometrical structures, as idealized constructions, can correspond only approximately to empirical contents presently under consideration. Possibly different (e.g., non-Euclidean) interpretations of the same phenomena cannot be excluded" (p. 52).
8. Note, however, that in 1878 during discussion with Albrecht Krause, Helmholtz uttered the famous phrase that "space can be transcendental without the axioms being so." See the paper by Biagioli (2013) for further analysis.
9. Helmholtz was not alone in the claim that there is nothing special in Euclidean geometry. Hans Hahn (1980), the Austrian mathematician and member of famous Vienna Circle, stated that:

   > If the use of multi-dimensional and non-Euclidean geometries for the ordering of our experience continues to prove itself so that we become more and more accustomed to dealing with these logical constructs; if they penetrate into the curriculum of the schools; if we, so to speak, learn them at our mother's knee, as we now learn three-dimensional Euclidean geometry, then nobody will think of saying that these geometries are contrary to intuition. They will be considered as deserving of intuitive status as three-dimensional Euclidean geometry is today. For it is not true, as Kant urged, that intuition is a pure a priori means of knowledge, but rather that it is force of habit rooted in psychological inertia. (p. 101)

10. To make the following reconstruction of the Piaget's and collaborators views on mental construction of Euclidean space accessible, I intentionally skip the Piagetian nomenclature of developmental stages and substages. Instead, I list the approximate age at which, according to these researchers, children reach particular spatial capacities. All the skipped details can be found in the cited original works as well as in literature reviews. Note also that in the introduction to Piaget's theory of spatial development and its critique I frequently refer to the review by Clements and Battista (1992). A book by Roth (2011) is also a very useful position.

11. Note that the listed properties called by Piaget and Inhelder (1967) "topological" were investigated earlier by Gestalt psychologists. The child at the earliest developmental stage does not show, however, all the perceptual capacities known from Gestalt theory. Furthermore, not all of them can be understood, in Piagetian terms, as topological. As Piaget and Inhelder note "in opposition to the main hypothesis of the Gestalt theory, we believe (…) that perception of 'good configurations' (or simple Euclidean forms) itself evolves with age as a result of sensori-motor activity. Eye movements, tactile exploration, imitative analysis, active transpositions, etc., all play a fundamental part in this development" (p. 10).
12. Noteworthy, we will not find "connectedness" on Piaget and Inhelder's (1967) list of topological properties. Martin (1976a) declares, however, that "connectedness was selected for study because of its relative importance in topology and because, as a concept, it bears similarities to what appears to be Piaget's notion of continuity" (p. 29).
13. One should bear in mind that details of the van Hieles' model, especially the number of levels and their naming and numbering, have changed over time. Note that originally the levels were not numbered from 1 but started from 0. In the following reconstruction, I refer to the works by Clements and Battista (1992) and Battista (2007).
14. The same is true about the perspective on cognitive development by Vygotsky (1934/1986), which will be introduced in the next chapters.
15. Note that 1956 was not only the time of the aforementioned symposium, but also the publication of several works, for example, by Bruner, Goodnow, and Austin (1956); G. A. Miller (1956); Newell and Simon (1956); or Shannon and McCarthy (1956), which rapidly became classics for the cognitive science.
16. Studies on geometric theorem-proving programs were "in touch" not only with psychological analysis of verbal protocols, but also had educational recourses. For instance, when discussing the DC model, Koedinger and J. R. Anderson (1990) claimed that "the organization of knowledge in DC suggests an alternative task-adapted organization of the geometry curriculum. Typical geometry curricula are organized around topics, and focus on teaching the formal rules of geometry. Alternatively, a curriculum could be organized around diagram configuration schemas (…). The formal rules, then, could be taught in the context of how they are used to prove schemas. Such a task-adapted curriculum organization can help students remember rules and access them in the appropriate situations" (pp. 547–548).
17. It is noteworthy that philosophical sources of embodiment, as well as the birth of the term itself, go back to phenomenological tradition, for instance, Merleau-Ponty's works (e.g., 1945/2002). Psychological sources are also much older than the 1980s. Arguably, since Piaget (1926) emphasized the pivotal role of bodily activity, that is, exploration of the surroundings, manipulation of objects, and internalization of these actions, in constructing cognitive structures, he may also be considered one of the predecessors of the embodiment (see Marshall, 2016). On the other hand, according to Piaget's approach, the role of the bodily activity decreases with age, and when the child reaches so-called formal operation stage, her thinking becomes abstract and initial embodied grounding of cognition fades. This idea contrasts with embodied cognitive science: its theorists claim that perceptual and motor activity constitute cognition across one's lifespan.

18. Note that among the users of right-to-left reading systems, for example, Palestinians, the reverse SNARC effect has been revealed. Namely, they respond faster for small numbers with their right hand and for large numbers with their left. In the case of Israelis—who read the Hebrew language from right to left, but Arabic numbers in the reverse order—no spatial-numerical associations reflected in RTs were initially found (Shaki, Fischer, & Petrusic, 2009). More recently, however, Zohar-Shai, Tzelgov, Karni and Rubinsten (2017) revealed the SNARC effect in native Hebrew speakers.
19. The observant reader will have noticed that my previous book, written together with my colleague Bartosz Brożek in Polish, is present in the set of literature references. I should confess that its title (in English, *The Mathematical Mind*) is misleading, since it promises more than we could actually give. Besides a few references to geometry, the book is dedicated entirely to the cognitive processing of numbers and some broader theoretical issues, however, associated mainly with numerical processing.

1. I could call these "hardwired" capacities "the geometric sense" in reference to "the number sense," a term introduced by Dantzig (1954; see Berch, 2005 for discussion) and popularized in contemporary studies on number processing by Dehaene (2011; see Section 1.6 of Chapter 1). Instead, although it is perhaps just a matter of semantics, I prefer to talk about "hardwired geometric skills" (or capacities) enabling "hardwired geometric cognition." I have also adopted such a convention in an earlier article (Hohol & Miłkowski, 2019). The main reason is that the current state of research indicates that there is no single mental system that makes knowledge about geometry possible.
2. It should be noted that younger children cannot be tested in this reorientation task since it requires independent locomotion.
3. Note that Mayr (1961) proposed a distinction between proximate and ultimate factors that correspond to different aspects of scientific explanation. While the former refers directly to the factors underlying explaining phenomenon, the latter concerns the evolutionary origin of the phenomenon. While Tinbergen's (1963) questions (1) and (2) correspond to Mayr's proximate factors, items (3) and (4) are close to ultimate ones (see also Table 2.1).
4. Prinz (2006), for instance, noted that "systems that have been alleged to be modular cannot be characterized by the properties on Fodor's list. At best, these systems have components that satisfy some of Fodor's criteria. There is little reason to think that these criteria hang together, and, when considered individually, they apply to a scattered and sundry assortment of subsystems. It is grossly misleading to say that the mind is modular. At best, the mind has a smattering of modular parts" (p. 32).
5. Note that the parahippocampal place area, a structure that has been perceived as directly involved in spatial navigation, recently turned out to be insensitive to sense (left-right) information (Persichetti & Dilks, 2016). The results of a recent neuroimaging study by Dillon and colleagues (2017) show that this structure is sensitive to relative length and angle but only in the case of pictures depicting scenes (not objects). This finding demonstrated that the main role of this structure, as the authors note, "may not be for navigation through a scene, but rather for scene categorization (e.g., recognizing a place as a kitchen or beach), consistent with the classic evidence that shape analysis is central to object recognition and categorization" (p. 8).
6. Many advocates of evolutionary psychology, such as Pinker (2009) and Buss (2009), seem to reject this claim. Instead, they accept the adaptive lag hypothesis, which states that the shape of contemporary cognitive systems is the result of adaptation to

Pleistocene—not contemporary—environmental conditions and challenges. As Cosmides and Tooby (1997) explicitly say, "our modern skulls house a stone age mind" (p. 10). This hypothesis, however, has been criticized by many authors (see Buller, 2006; E. A. Smith et al., 2001). The necessity of studying "current utility" is in line with a crucial assumption of human behavioral ecology that "human beings are able to alter their behavior flexibly in response to environmental conditions in a manner that optimizes their lifetime reproductive success" (Laland & G. R. Brown, 2006, p. 93).

7. In this context it is worth mentioning the Artificial Life simulations by Ponticorvo and Miglino (2009). They find that "different orientation abilities can emerge, varying systematically the exposure to different environmental cues. It is possible to evolve agents with different spatial skills by varying the frequency with which they are exposed to different classes of stimuli during their evolution. Agents that evolve in environments providing balanced exposure to geometric and non-geometric cues acquire the ability to use both kinds of clue. Agents that are exposed primarily to a single class of cue show primacy. This supports our hypothesis, according to which geometric primacy, non-geometric primacy or successful integration between the two classes of information depend on the relative frequencies at which organisms are exposed to these information during their evolution and development" (p. 170).
8. Note that current knowledge about the core geometric cognition of nonhuman primates is very limited. For example, the geometry-based navigation abilities of our closest relatives, that is, bonobos and chimpanzees, are—to the best of my knowledge—so far completely unexplored.
9. Hermer-Vazquez and colleagues (2001) treated children's performance in a reorientation task with a landmark as a dependent variable, while the tested independent variables involved, among other things, age, fluid intelligence, digit span, spatial memory span, and the comprehension and production of various spatial phrases.

1. It is noteworthy that while a "mental representation" is a basic notion in both classic cognitive science and most of its contemporary variations (see, however, note 8 in this chapter), there is still debate as to what really deserves to be called full-blooded representation. This question goes hand in hand with the charge that the notion is used in interdisciplinary studies on cognition—that is, in the case of simple detectors of features—in a too liberal manner. The reader interested in this topic should reach for Ramsey's (2007) book and more recent articles by Gładziejewski (2015) and Thomson and Piccinini (2018).
2. Note, however, that this is an interpretation of the initial version of Kosslyn's approach. The theory of mental imagery has changed over time, among others under the pressure of neuroscientific data, but I do not have the opportunity to trace its development here (see Ganis, Thompson, & Kosslyn, 2004; Kosslyn, 1996; Kosslyn, Ganis, & Thompson, 2003; Pearson & Kosslyn, 2015).
3. Clever Hans was an Orlov Trotter horse owned and trained by a Berlin-based mathematics schoolteacher, Wilhelm von Osten (1838–1909). The latter was firmly convinced that Hans manifested mathematical capacities and other humanlike highly intellectual skills. At the beginning of the twentieth century, the alleged talents of Clever Hans were investigated by Carl Stumpf, who was one of the founding fathers of German experimental psychology, and his assistant Oskar Pfungst. It turned out that the intriguing behavior of the horse was a response to bodily cues delivered, in an unconscious and involuntary way, by the owner (see Dehaene, 2011, pp. 4–7, and Samhita & Gross, 2013 for more details and further discussion).

4. On the other hand, although earlier philosophical empiricists definitely rejected nativism, the theory of perceptual symbols does not seem to exclude the innateness (or hardwiredness, in the terms adopted in the book) of some components of mental equipment (Markman & Dietrich, 1999).
5. This does not mean, however, that we cannot point the defenders of the amodal approach to cognition. Pylyshyn (1973; 1981), for instance, argued fervently against the cognitive function of mental images.
6. Note, however, that in contrary to J. J. Gibson (1979/2015), who claimed that affordances are perceived directly by the observer ("directly," means without the need for forming mental representations), Glenberg and Robertson (1999) explicitly "allow for the mental representation of affordances" (p. 4). It should be mentioned here that Gibson's ecological psychology inspired antirepresentationalist cognitive science (see also note 8 in this chapter).
7. So far in the domain of experimental psychology of mathematics, only a few studies with the participation of professional mathematicians have been carried out. In addition to those of Amalric and Dehaene quoted in this paragraph, here are a number of others: Cipora, Hohol, Nuerk, Willmes, Brożek, Kucharzyk, & Nęcka, 2015; Sella, Sader, Lolliot, & Cohen Kadosh, 2016; Zeki, Romaya, Benincasa, & Atiyah, 2014.
8. Note that the existing terminology is fuzzy and there are also various ways of characterizing variants of embodiment. For instance, wide-scope as well as narrow-scope embodied cognition in Machery's (2007) sense, introduced in Section 3.3, could be considered moderate embodiment since both perspectives preserve a standard view of cognitive psychology (resp. cognitive science) that cognitive activity consists of processing the mental representations. The controversy concerns the format of representations (modal-amodal) but not their existence (see, e.g., van Elk, Slors, & Bekkering, 2010). In contrast, the radical theories of embodiment, such as those proposed by Chemero (2011) and Gallagher (2017), reject a notion of mental representation entirely. To the best of my knowledge, there are no empirically fruitful accounts of geometric cognition within the radical embodiment (in the above meaning), and thus I will not discuss this perspective (but see Hutto, 2019; Roth, 2011 for the attempt to adopt an antirepresentational attitude to mathematical thinking in general).
9. Note that the term "scaffolding" was introduced by Jerome Bruner in the middle 1970s (see D. Wood, Bruner, & Ross, 1976); however, it was strongly influenced by Vygotsky's work (see also Lajoie, 2005; Sterelny, 2010).
10. Linguistic arbitrariness can be defined as "the unpredictable mapping of form and meaning such that, apart from a social convention to use word A for meaning B, there is no connection between the sound of a word and aspects of its meaning" (Dingemanse, Blasi, Lupyan, Christiansen, & Monaghan, 2015, p. 604).
11. Dove (2011) called his approach dis-embodiment, wherein the dash is crucial, due to the distinction from amodal theories. According to him, "A mental symbol is dis-embodied if (1) it is embodied but (2) this embodiment is arbitrarily related to its semantic content. In other words, a mental symbol is dis-embodied if it involves sensorimotor simulations of experiences that are not associated with its semantic content" (p. 6).

1. Of course, mathematicians and philosophers also attribute other features to mathematical proof. They frequently say, for instance, that a good proof should be economical (Hardy, 1940/2005, p. 29), explanatory (Mancosu, Jørgensen, & Pedersen, 2006), and even beautiful (Heller, 2012; Rota, 1997).

2. Note that Netz explicitly identifies "universalist" cognitive science with Fodor's (1983) modular theory of mind. Although the characteristics of core systems of geometry (Spelke, S. A. Lee, & Izard, 2010) cannot be directly translated into a Fodorian account, as we have seen in Chapter 2, these systems, similarly to modules, are initially (namely in early childhood) universal to all human individuals and shared with nonhuman animal species. Hence, Netz would probably agree that cognitive science focused on core systems can also be called "universal."
3. A distinction between the vertical and horizontal cultural transmission of knowledge has been introduced by Cavalli-Sforza and Feldman (1981).
4. In Sections 4.2–4.4, I refer to our investigation presented in the article entitled *Cognitive Artifacts for Geometric Reasoning* (Hohol & Miłkowski, 2019).
5. Hilbert's axioms refer to the properties involving incidence (8 axioms), order (4 axioms; the Pasch axiom is one of them), congruence (6 axioms), parallels (1 axiom) as well as continuity (2 axioms).
6. This is illustrated by Hilbert's famous dictum: "One must be able to say at all times– instead of points, straight lines, and planes—tables, chairs, and beer mugs" (Corry, 2004, p. 124).
7. Note that our knowledge about the functions of Greek diagrams was, until recently, incomplete. As Netz noted, "the scholars who edited mathematical texts in the nineteenth century were so interested in the *words* that they ignored the *images*. If you open an edition from that era, the diagrams you find are not based upon what is actually drawn in the original manuscripts. The diagrams represent, instead, the editor's own drawing" (Netz & Noel, 2009, p. 31). Detailed information on this matter can be found in work by Saito and Sidoli (2012).
8. Historians of mathematics, or historians of science, in general, frequently highlighted the existence of "the genetic link between rhetoric and the hypothetico-deductive method," to refer to Russo's (2004, p. 196) words. Furthermore, as the researcher suggests, "the scientific method, too, had its roots in oral culture, thus going back to long before the Hellenistic period" (ibid.). There is also agreement that orality played a very important role in Greek culture (see Thomas, 1992).
9. We critically discuss Netz's thesis of the substitution of mathematical ontology by diagrams in Hohol and Miłkowski (2019). According to our interpretation, Netz's thesis implies that diagrams, understood as cognitive artifacts, or cognitive tools (in his terminology), are sufficient for not only mathematical practice, but also that *no* semantic considerations are needed (see Latour, 2008). We propose that the thesis about the neutrality of diagrams should be considered as being divided into two components, namely ontological and epistemological ones. We claim that the use of diagrams (and linguistic formulae as well) is ontologically neutral, but simultaneously, there is no epistemological neutrality connected with using them. While diagrams considerably constrain the permitted steps when one conducts the geometric reasoning (proof), the issue of mathematical ontology assumed by her is still open. In other words, diagrams do not constrain ontological choices. From the point of view of mathematical practice (or in the context of justification), it is not crucial whether the geometer accepts the realistic (i.e. Platonic or Aristotelian) interpretation of geometric constructions (where a diagram serves as a model of the eternal geometric object) or Menaechmus' constructivist approach (where there is nothing "outside" a geometric construction that is made by the human being). The acceptance of a particular philosophical option may be important, however, in the context of mathematical discoveries (Reichenbach, 1938; see also Giaquinto, 1992). It is

noteworthy that the textual evidence does not exclude the fact that the purely empirical nature of the earliest measurement of the Earth developed by the Egyptians stimulated a realistic rather than a constructivist attitude on the part of Greek mathematicians, despite the transformation of geometry into a science of abstract concepts. This hypothesis is consistent with the fact that Greek geometry has not only a purely mathematical face, but also a practical one (Russo, 2004). The use of geometry in Greek architecture is an example that immediately comes to mind (Leonardis, 2016). The use of physical tools could also stimulate a realistic attitude. In this context, it is worth mentioning A. Seidenberg's (1959) hypothesis that the idea of geometric proof emerged from constructions made by using a peg and cord. To sum up this philosophical digression, cognitive artifacts indeed play the constraining role in the context of mathematical justification. Thanks to them, epistemological practice is completely isolated from questions such as "does the geometric point, namely something 'that of which there is no part' (Definition 1, Book 1 of Euclid's *Elements*) exist in reality"? Regardless of how the geometer answers this question, the proof remains correct. This is possible because the space of admissible operations when we prove the theory is constrained by the properties of publicly shared cognitive artifacts. These properties do not determine, however, the ontological status of geometric objects. Our thesis implies that the necessity and generality of Euclidean geometry are achieved in an ontologically neutral way, because these epistemic virtues emerge from the use of cognitive artifacts.

10. Regarding the acquisition of formulae, Lord (1960) stated that, "When we speak a language, our native language, we do not repeat words and phrases that we have memorized consciously, but the words and sentences emerge from habitual usage. This is true of the singer of tales working in his specialized grammar. He does not 'memorize' formulas, any more than we as children 'memorize' language. He learns them by hearing them in other singers' songs, and by habitual usage they become part of his singing as well. Memorization is a conscious act of making one's own, and repeating, something that one regards as fixed and not one's own. The learning of an oral poetic language follows the same principles as the learning of language itself, not by the conscious schematization of elementary grammars but by the natural oral method" (p. 36).